Product Differentiation in Two-Sided Markets

PhD Dissertation

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Um dia o amor, intrigado, virou-se para a amizade e perguntou-lhe:

-"Ó amizade, porque é que tu existes ... se já existo eu ?"

A amizade respondeu-lhe:

-"Para repor um sorriso ... onde tu podes ... deixar uma lágrima ..."

Para todas as pessoas que me amam, serei para sempre vosso amigo.

(VMR, 2014)

Biographical Note

Vitor Miguel Ribeiro was born on the 28^{th} of September of 1986 in the region of Canelas, Vila Nova de Gaia in the north of Portugal.

In 2004, he entered in the Faculdade de Economia da Universidade do Porto, Portugal. During his first academic year, he completed only one course (Mathematics, with a final classification of 14 in 20 only) but without attending any classes since he went to the city of Bilbao in the north of Spain, where he worked the entire year as a construction worker. Since 2005, he has dedicate himself exclusively to his undergraduate course in Economics although periodically working as a representative salesman. In 2009, he completed his undergraduate degree in Economics, with a final average classification of 15.4 (in 20).

He has been enrolled in the PhD Program of Economics at the Faculdade de Economia da Universidade do Porto since October 2009. With a grant provided by the FCT (Fundação para a Ciência e Tecnologia), he has dedicated himself exclusively to the PhD Program. He completed the theoretical part of the PhD Program in June 2011 with a final average classification of 17 (in 20). Having the need to teach (besides being an eternal learner), he interrupted the PhD program for six months to be a lecturer in Sales and Logistics, teaching professionals of the retail sector. Then, he returned to the PhD Program. He spent six months working on an unsuccessful manuscript and then decided to interrupt the PhD program for six months again. In this second stop, he worked for Galp Energia where he was responsible for performing financial analysis (in the projects "Ventinveste, SA" and "Biofuels"), determination of the multinational WACC and econometric analysis (in the projects "Ventinveste, SA" and "Reconversão do ciclo de input-output nas refinarias de Sines e Matosinhos"). After March 2012, he dedicated himself exclusively to his thesis. During this period, he completed the four chapters that compose his thesis and wrote five additional papers, attended to international conferences (UECE in Lisbon, Oligo Workshop in Rome, Annual Meeting of the PEJ in Braga) and seminars (FEP.UP in Porto) and he was a referee for some international journals. His main scientific areas of research are Industrial Economics, Behavioral Economics, Regional Economics and Environmental Economics.

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Lastly, I thank to the potential recruiters that will read my thesis and may give me a job opportunity (hope so!). Besides this thesis, it is with pleasure that I am completing five other manuscripts which I did not incorporate them here. I did my best efforts to differentiate myself from other PhD candidates in terms of quality and variety. I love to teach and to learn, I love Economics and I wish that this last five years will provide me some kind of compensation in the future. Finally, I thank the members of the jury Cesaltina Pires, Duarte Brito, Hélder Vasconcelos and Paul Belleflamme for their important and helpful comments that allowed me to improve the quality of this dissertation.

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Resumo

Esta tese estuda um mercado bilateral com a presença de diferenciação de produto horizontal e vertical. Baseamo-nos em Armstrong (2006) [9] segundo o qual duas plataformas horizontalmente diferenciadas concorrem em preços. A procura é composta por dois grupos simétricos de "singlehoming" agentes que beneficiam de uma externalidade. A diferenciação vertical é introduzida de três formas (respectivamente, nos capítulos 3, 4 e 5).

No capítulo 3, a diferenciação vertical consiste numa densidade de consumidores variando ao longo da linha unitária, tal como em Gabszewicz e Wauthy (2012) [57]. Em equilíbrio, a plataforma de alta qualidade cobra um preço mais elevado e capta uma quota de mercado superior relativamente à rival. O equilíbrio socialmente ótimo requer maior assimetria entre as quotas de mercado das plataformas. Uma alteração na densidade de consumidores no lado esquerdo relativamente ao lado direito do centro da cidade perturba a existência de equilíbrio tal como descrito por Armstrong (2006) [9].

No capítulo 4, consideramos uma assimetria na utilidade de reserva das plataformas. Os preços de equilíbrio, quotas de mercado e os lucros da plataforma de baixa qualidade (de alta qualidade) estão a diminuir (aumentar) com o grau de diferenciação vertical. Analisamos também um jogo de dois estágios em que as plataformas investem para aumentar os seus níveis de qualidade e, em seguida, competem em preços. A plataforma com custo de investimento mais baixo torna-se o líder de mercado, mesmo cobrando um preço mais elevado. Numa variação do modelo em que o custo de investimento difere entre os dois lados do mercado, estratégias "dividir para conquistar" surgem em equilíbrio. A plataforma de baixa qualidade cobra um preço menorque o seu custo marginal no lado onde a plataforma de alta qualidade tem um menor custo de provisão de qualidade.

No capítulo 5, a diferenciação vertical é introduzida *à la* Neven e Thisse (1990) [87]. O equilíbrio depende da força da externalidade vis-à-vis magnitude das diferenças de qualidade entre as plataformas. Se a asssimetria de qualidade é suficientemente pequena, o lucro da plataforma de baixa qualidade diminui à medida que a qualidade da rival aumenta. Um incremento na externalidade tem um impacto negativo sobre os preços e lucros de equilíbrio.

Abstract

This thesis studies two-sided market duopolies with simultaneous horizontal and vertical product differentiation. It builds on the model of Armstrong (2006) [9] in which two horizontally differentiated platforms engage in price competition. Demand is composed by two symmetric groups of agents that singlehome and benefit from an inter-group externality. Vertical differentiation is introduced in three different ways (resp., in chapters 3, 4 and 5).

In chapter 3, vertical differentiation consists of a varying consumer density along the unit line, as proposed by Gabszewicz and Wauthy (2012) [57]. We find that, in equilibrium, the high-quality platform charges a higher price and captures a greater market share than the low-quality platform. As a result of the inter-group externality, the socially optimal outcome requires a greater asymmetry between firms' market shares. A perturbation that introduces a small difference between the consumer density on the left and on the right side of the city may disrupt the existence of equilibrium in the model of Armstrong (2006) [9].

In chapter 4, vertical differentiation is modelled as an asymmetry in the stand-alone values of the two platforms. The equilibrium prices, market shares and profits of the low-quality (high-quality) platform are decreasing (increasing) in the degree of vertical differentiation. We also analyse a two-stage game in which platforms start by investing to increase their quality levels and then compete in prices. The platform with lower investment cost becomes the market leader even though it charges a higher price. We also study a variation in which the investment cost differs across sides. We find that divide and conquer strategies arise in equilibrium. The low-quality platform charges a price below its marginal cost in the side where the high-quality platform has a lower cost of quality provision.

In chapter 5, vertical differentiation is introduced *à la* Neven and Thisse (1990) [87]. We find that equilibrium outcomes depend on the strength of the inter-group network effects vis-à-vis the magnitude of the intrinsic quality differences between the platforms. If the quality asymmetry is sufficiently small, the profit of the low-quality platform decreases as the quality of the better platform improves. We also conclude that the intensity of the inter-group externality has a negative impact on equilibrium prices and profits.

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1.0 INTRODUCTION

The goal of this thesis is to provide an analysis of platforms' pricing decisions dealing simultaneously with vertical and horizontal differentiation in a two-sided market.

Rochet and Tirole (2003) [94] define a market as two-sided if "*platforms serve two distinct groups of agents, such that the participation of at least one group raises/decreases the value of participating for the other group*". Product differentiation is a field that attracts considerable attention since Lancaster's (1980) [80] definition of product space.

In the Industrial Economics literature, two types of product differentiation can be distinguished: horizontal differentiation (variety) and vertical differentiation (quality).

The majority of the research related to horizontal differentiation is based on Hotelling (1929) [68] and his principle of minimum differentiation and in d'Aspremont, Gabszewicz, and Thisse (1979) [28] and their principle of maximum differentiation.

On the literature of vertical differentiation (see, for example, Mussa and Rosen (1978) [85], Gabszewicz and Thisse (1979) [54], Shaked and Sutton (1982) [105]) there appears an equilibrium where firms are placed at opposite ends of the quality spectrum. Moorthy [83] extends the model by incorporating marginal costs and allowing for partial market coverage.

Much research extended the unique dimension of product differentiation into several dimensions (e.g., de Palma et al. (1985) [32], Neven and Thisse (1990) [87], Economides (1989) [39] and Economides (1996) [41].

De Palma et al. (1985) [32] finds that the principle of minimum differentiation holds when "*products and consumers are sufficiently heterogeneous*". Economides (1989) [39] and Neven and Thisse (1990) [87] analyze a two-dimensional vertical and horizontal differentiation model in which firms compete on quality, variety and price.

Economides (1989) [39] assumes that the horizontal choice takes place before the vertical choice and that marginal costs are increasing in quality. He finds the prevalence of maximum horizontal differentiation and minimum vertical differentiation. In Neven and Thisse (1990) [87], firms first choose their location and subsequently choose their prices. Disregarding any asymmetries between the firms, they find a product equilibrium that exhibits maximum differentiation in one dimension and minimum differentiation in the other. However, the maximally differentiated dimension can be either the horizontal or the vertical one.

Other contributions such as Irmen and Thisse (1998) [69], Daughety and Reinganum (2007) [29], Daughety and Reinganum (2008) [30], Argenziano (2008) [7], Levin, Peck and Ye (2009) [82], Griva and Vettas (2011) [63], Gabszewicz and Wauthy (2012) [57] and Amir, Gabszewicz and Resende (2013) [4] conduct studies on simultaneous vertical and horizontal differentiation (the last article embraces it in two-sided markets, however, not taking into consideration the existence of access or transaction prices).

This thesis relies on an analytical toolbox based in Game Theory. The methodology is developed in three phases: in a first phase we present the models who study the behavior of all the players involved using mathematical tools; on a second phase we determine the equilibrium outcomes in a static game with price competition. Finally, in a third phase we display the results, conclusions and policy implications by analyzing the respective effects on the consumers' and platforms' behavior, namely in terms of equilibrium pricing decisions, equilibrium number of agents on board, profitability and welfare analysis.

The thesis is organized in six distinct chapters: this first chapter corresponds to the *"Introduction"* and the next chapter performs *"A bibliometric analysis of two-sided markets"*. The central chapters of the thesis are chapters 3, 4 and 5, *"Nesting vertical and horizontal differentiation in two-sided markets"*, *"Endogenous quality and group discrimination in two-sided markets"* and *"Price competition between verti-zontally differentiated platforms"*, respectively. Finally, the conclusions of the thesis emerge in the sixth chapter. Although the chapters are related, each one constitutes an autonomous manuscript in terms of formulation of the theoretical setting, equilibrium computations and conclusions.

2.0 A BIBLIOMETRIC ANALYSIS OF TWO-SIDED MARKETS

Abstract. In this chapter, we provide a characterization of two-sided markets, expose a literature review and briefly cover the recent developments on the field under scrutiny. We also analyze quantitatively the field of two-sided markets over the past twelve years, proposing a categorization of a large dataset gathered from Scopus since 2002 to the end of 2013. The quantitative analysis concludes that the majority of two-sided markets articles are published in the *International Journal of Industrial Organization*, especially after 2010. Most of the research in this field is done in Europe and (North) America, and the authors that have more quantitative and qualitative relevance are Jean Charles Rochet, Jean Tirole and Mark Armstrong.

Keywords: Two-sided markets, bibliometrics.

JEL Classification Numbers: D40, L10, C89.

2.1 INTRODUCTION: DEFINITIONS AND CHARACTERISTICS

The debut of a general theory on two-sided markets has essentially three main causes: the widely spread concern with the antitrust cases against international credit card networks (Visa and MasterCard) in the United States, Europe and Australia around 2000 (Roson (2005) [97]), the growth of online markets and the questions concerning the respective regulation of these new emergent markets.

Several definitions are proposed in the literature to define a two sided-market. Rochet and Tirole (2003) [94] define a market as two-sided if "*platforms serve two distinct groups of agents, such that the participation of at least one group raises/decreases the value of participating for the other group*".

Roson (2005) [97] points out that "*two sided markets are defined as platforms providing goods and services to two distinct end users, where the platform attempts to set the price for each type of end user to get both sides on board because the benefits of one type of end user increases as the participation of the other type of end user increases*".

Rochet and Tirole (2006) [95] propose a more restrictive definition, since the term two-sided is only applied to "*cases in which prices faced by agents on each side (possibly zero or negative) have a direct influence on market participation for the other side, so that the volume of transactions does not merely depend on a comparison between total expected benefits and total transaction costs*". We propose a definition of a two sided market, as "*a market intermediated by a platform that allows two different but mutually attracted sides to connect in order to achieve a particular purpose*".

The platform is a physical or non-physical intermediary that may face competition or may hold a monopoly position.

As Rochet and Tirole (2003) [94] point out, "*the platforms may be classified as proprietary platforms meaning that they have profitable proposes*" or "*non-proprietary platforms (associations), meaning that they have no profitable purposes*". Some platforms' examples on the literature are: newspapers market and the media industry in general (see, for example, Anderson and Coate (2005) [5] and Correia-da-Silva and Resende (2013) [27]) or dating agencies market in the case of competing matchmakers (see, for example, Caillaud and Jullien (2003) [21]).

In the context of the literature on two-sided markets, a side is composed by a group of agents that only interact with other different type of agents through an intermediary. The platform may have only one member in each side or, in the limit, an infinite number of members. The wish for interaction between agents of different sides generates a positive network effect. Examples of sides are advertisers and readers in the case of newspapers, men and women in a club or a software developer and a end-user player in the case of game consoles.

Due to inter-group externalities, platforms using conventional pricing schemes may deal with the chicken-and-egg problem (Caillaud and Jullien (2003) [21])[1]. As Roson (2005) [97] mentions, models avoid this issue by assuming a rational expectations equilibrium, that is, the simultaneous arrival of agents on both sides of the market, which may not be consistent with the empirical evidence[2].

From a theoretical perspective, modeling a two-sided market is related with modeling markets embracing simple network effects, widely analyzed especially since the contributions by Katz and Shapiro (1985) [74], Farrell and Saloner (1986) [48], among others. In such context, simple network effects models were extended incorporating cross network effects between the sides of the market, which is one of the relevant characteristics of two-sided markets. Thus, inter-group externalities are present in two-sided markets since the participation of a certain side depends on the participation of agents of an opposite market side (see, for example, Rochet and Tirole (2003) [94]).

The literature on two-sided markets also considers, according to Belleflamme and Toulemonde (2009) [13], the concept of a positive (or negative) intra-group participation externality, that is, the participation of a certain side depends positively (or negatively) on the participation of agents of the same market side.

[1] The chicken and egg problem relies on the fact that to convince some buyers to adopt a certain platform, first it is necessary to persuade some sellers to join that platform. However, to convince the sellers there must be some buyers on the market, creating problems in benching the market.

[2] Roson (2005) [97] indicates that consoles (e.g., Playstation) to get user players must appear on the market already equipped with a huge range of games but to get the required technology, need to adopt pre-strategies, such as the under-cost provision of some games, free trials, among others which requires them to spend an extra-effort on the software developers side.

Table 1 below provides real world examples of two-sided markets[3].

Market	**Side 1**	**Side 2**	**Platform**
Operating systems	consumers	developers	Windows
Financial markets	firms	investors	Moody's
Online recruitment	unemployed	employers	EraCareers
Yellow Pages	consumers	advertisers	PAI (in Portugal)
Web search	searchers	advertisers	Google
Healthcare	patients	physicians	HMO
Video games	players	developers	Playstation
B2C	shoppers'	retailers	Shopping mall
Social networks and dating	men	women	Facebook
Media	viewers/readers	advertisers	New York Times

Table 1: Examples of two-sided markets

Another relevant aspect for the theory of two-sided markets is the issue of pricing. Rochet and Tirole (2006) [95] introduce a distinction between price level and price structure. Price level is defined as "*the total price charged by the platform to the two sides*" and price structure is defined as "*the decomposition or allocation of the total price between the buyer and the seller*".

According to Roson (2005) [97], "*the price structure gave rise to a notion: economic efficiency can be improved by charging more to one side and less to the other one, depending on the externalities and characteristics of the market*".

Within the concept of price structure, a platform may charge to both sides' transaction fees and/or access fees. A transaction fee is charged by the platform in each transaction between the sides and an access fee is charged by the platform only once to allow both sides to access the market.

Two-sided markets can present specific pricing schemes. For instance, when a platform charges transaction fees to one side of the market but pays to the other side (subsidization)

[3] Many examples of two-sided markets are provided in Rochet and Tirole (2003) [94], Eisenmann, Parker and Van Alstyne (2006) [43] and Armstrong (2006) [9].

to promote trading when the externality effect of the subsidized side is sufficiently strong, we say we turn to the "*topsy-turvy principle*" defined by Rochet and Tirole (2006) [95] where "*a factor that is conducive to a higher price in one side (such that it raises the platforms' margin in that side), tends also to call for a lower price in the other side (as attracting members on that other side becomes more profitable)*"[4].

Regarding the possibility of attending one or more platforms, the allocation of agents through the market is another fundamental characteristic enunciated by Armstrong (2006) [9]. Singlehoming is a concept that defines the situation where each agent on the market only attends to one platform. Singlehoming occurs, for instance, when a reader is loyal to a certain newspaper. Agents may also multihome[5]. In this case, to "impose" consumers' loyalty, platforms may steer. Steering is an action in which a platform does not allow their members to interact in an alternative platform. Clearly, the goal of steering is to promote singlehoming. Competitive bottleneck arises when one side multihomes and the other side of the market singlehomes (Armstrong (2006) [9]).

2.2 LITERATURE REVIEW

This section presents a brief review of the literature on two-sided markets. Firstly, we present the seminal contributions that constitute the genesis of this field of research. Then, considering an overall perspective, we present more recent contributions to the literature.

[4]The topsy-turvy principle is also connected with recent discussions towards the concept of freemium in online markets (see Lambrecht et al. (2012) [79]), cross subsidization (see Chen and Rey (2013) [26]) and behavior-based price discrimination (Fudenberg and Villas-Boas (2006) [53]). Note that, a divide and conquer strategy corresponds to an application of this principle in two-sided markets.

[5]Multihoming defines the possibility of both sides to interact through several platforms. For instance, consumers multihome when they own several credit cards, advertisers multihome using several newspapers to advertise, people may decide to participate on several social networks, etc. Multihoming is more easily observed when the access fees charged by a platform are low or null (see, for example, Kaiser and Wright (2006) [72]).

2.2.1 Seminal contributions

The seminal contributions on two-sided markets are provided by Armstrong (2006) [9] as a general model of two-sided markets, by Rochet and Tirole (2003) [94] especially focusing the debit/credit card markets and by Caillaud and Jullien (2003) [21] for an analytical treatment of two sided-markets internet particularities. We briefly present their main conclusions.

Rochet and Tirole (2003) [94] build a model for a two-sided market in order to unveil the determinants of price allocation and end-users surplus for different governance structures in the case of an integrated monopolist and under duopoly competition.

The authors state that under symmetric information, the price structure of an integrated monopolist is given by the ratio of the elasticities:

$$\frac{p^s}{\eta^s} = \frac{p^b}{\eta^b},$$

where p^i is the price charged by the monopoly platform to side i and η^i is the quasi-demand price elasticity of side $i, i = b, s$ (buyers and sellers, respectively).

The Ramsey price is the resulting price that arises from achieving the efficient outcome that allows the monopolist platform to maximize the social welfare subject to some constraints. The authors find that the outcome that maximizes welfare is the same that maximizes the monopoly platform's profits.

Considering competition between platforms, the authors analyze the impacts on the sellers' side of a decreasing fee charged by a cheaper platform, keeping constant the fee charged by an expensive platform. In this context, the authors conclude that the seller has three options: no trade, if the seller's surplus is lower than the price charged by the cheapest platform; interact with buyers only in the cheapest platform, if the seller's surplus is higher than the price charged by the cheapest platform but the seller's surplus is lower than an "indifferent surplus" between trading in both platforms or interact with buyers in both platforms, if their surplus is higher than an "indifferent surplus" between trading in both platforms.

The existence (or not) of steering allows to evaluate the market power of a certain platform. The authors take into account the distinction between competition of proprietary platforms (with profitable proposes) and associations (with no profitable proposes) and conclude that "*even when downstream markets are perfectly competitive and the price level is socially optimal, competition between associations does not generate an efficient outcome*".

Finally, the authors evaluate the effect of the transaction fee on the volume of transactions, given the presence of three features: marquee users[6], installed bases (or captive users) and multihoming[7]. Marquee buyers (sellers) are a type of buyer (seller) that generates higher surplus on their respective side since they are considered as extremely valuable. Under competition or under a monopoly (and considering a profitable platform or an association) and with log-concave demand functions, the authors find that the seller price increases when there are marquee buyers on the market and decreases with the presence of captive buyers. On the other hand, the buyer price moves in the opposite direction.

Armstrong (2006) [9] suggests three possible models for two-sided networks: a monopoly platform, a model of competing platforms where agents join a single platform (pure singlehoming) and a model of "competitive bottleneck" where one group joins all platforms (partial multihoming). In the case of a monopoly platform, Armstrong (2006) [9] concludes that even when a certain side pays a lower fee relatively to its marginal cost, the platform might hold its market position since the monopoly platform can charge a high enough fee on the opposite side.

In the case of duopoly competition with pure singlehoming agents, Armstrong (2006) [9] adopts Hotelling (1929) [68] by assuming that each agent joins only one platform. The equilibrium outcome shows the prevalence of an extra effect with competition (the transportation cost t), which measures the degree of horizontal differentiation between the platforms. When price discrimination between sides is allowed, the author shows that the side that pays a higher fee gets a lower equilibrium surplus and concludes that an increment on the intensity of the inter-group externality is detrimental to the intermediaries' profit.

Under partial multihoming, the author concludes that a market failure exists in the sense that the interests of the multihoming side are ignored. However, as Armstrong (2006) [9] points out, at least three limitations in the analysis of competitive bottleneck are present:

[6] Marquee users may be buyers or sellers.

[7] The authors find that an increase in the multihoming index of buyers leads to an increase in the buyer price and a decrease in the seller price, caeteris paribus.

firstly, the population of the singlehoming side is constant (even when this group is treated favorably in equilibrium), secondly, the population of the singlehoming side never multihomes (however, for instance, people may decide to participate on several social networks) and lastly, Armstrong does not consider a platform's incentive to require an otherwise multihoming agent to deal with it exclusively.

Caillaud and Jullien (2003) [21] concentrate on two-sided markets inspired by online intermediation. The authors study peculiar features of two-sided markets: the presence of indirect network externalities, the possibility of using the non-exclusive services of several intermediaries at the same time and the widespread practice of price discrimination based on users' identity and usage. The authors focus their analysis in a matchmaking monopoly intermediary for dating services.

Using linear demand and a Bertrand pricing model, they explain why users register with more than one service and how, under competition, two-sided network effects can lead to several possibilities: (i) one platform may promote corner solutions; (ii) multiple platforms may share the market with zero profits. In the case where all agents singlehome, platforms face perfect competition and, consequently, the efficient outcome is achievable when agents use the same platform. In the case where one side multihomes the authors find that there is a mixed equilibrium, which corresponds to the case of competitive bottleneck defined by Armstrong (2006) [9].

2.2.2 A general overview of recent developments

The development of a general theory of two-sided markets brought the need for some theoretical refinements (e.g., Evans and Schmalensee (2005) [47], Eisenmann, Parker and Alstyne (2006) [43], Eisenmann, Parker and Alstyne (2011a) [44] and Eisenmann, Parker and Alstyne (2011b) [45]).

Evans and Schmalensee (2005) [47] provide general features that we can find in many two sided-markets. The first feature is related with "*getting both sides on board*". Here, the authors highlight that the intermediary must take actions to allow the trade between both sides. The second feature is related with defining an optimal price structure in order to carefully balance the two sides' demands. So, platforms must perform the balancing work between the two sides demand and often regulate the terms of the transactions between end-agents. The third idea is that multihoming affects both the price level and the price structure. According to the authors, "*the fee level tends to be lower with multihoming, since the availability of substitute platforms ensures higher pressure on the incumbent platform*" (Evans and Schmalensee (2005) [47]).

Finally, the authors mention that monopolies are not the ideal structures for a two-sided market, since "*the customers' heterogeneous preferences on either side encourage platform differentiation. Additionally, platform differentiation, coupled with the low level of switching costs, results in multi-homing, which in turn, provides demand for several platforms by customers*" (Evans and Schmalensee (2005) [47]).

Eisenmann, Parker and Alstyne (2006) [43] reinforce that platforms adopt a specific pricing strategy in which they have to subsidize one side of the market by setting a very low fee, often equal to zero or even negative (subsidy), precisely a similar idea to the one proposed by Armstrong (2006) [9] and Caillaund and Jullien (2003) [21]. They argue that "*a platform provider can increase its growth if it can commit marquee users to use its platform without joining rival platforms*" (Eisenmann, Parker and Alstyne (2006) [43]). This idea turned to be on the basis for the adoption of an exclusivity strategy by the platforms.

The same authors also give an important contribution when they mention that platforms have overlapping user bases and, thus, it is not uncommon for a platform to be "*enveloped*"[8] by an adjacent provider, especially when a rival platform provides the same functionalities (see Eisenmann, Parker and Alstyne (2011b) [45]).

The authors consider that, when a platform perceives that (in a multiplatform environment) a rival delivers more intrinsic quality at a lower price, its stand-alone value is in danger. On the other hand, if the platform is not able to reduce the fee charged or is not able to enhance their stand-alone value, the authors propose one of the following strategies to

[8] In their own words, the envelopment of a platform is defined by the fact that "platform providers that serve different markets sometimes have overlapping user bases and employ similar components. Envelopment entails entry by one platform provider into another's market by bundling its own platform's functionality with that of the target's so as to leverage shared user relationships and common components. Dominant firms that otherwise are sheltered from entry by standalone rivals due to strong network effects and high switching costs can be vulnerable to an adjacent platform provider's envelopment attack".(see Eisenmann, Parker and Van Alstyne, 2011a [44]).

overcome this issue: or to change the platform business model or to find a "bigger brother" to help it. In the limit, another last option when platforms' face envelopment is to resort to legal remedies, since the antitrust laws for two-sided markets are still subject of a huge controversy (see Eisenmann, Parker and Alstyne (2011a) [44] and Eisenmann, Parker and Alstyne (2011b) [45]).

The theoretical literature on two-sided markets has improved our understanding of a wide range of economic problems, in particular, the definition of an optimal price structure in two-sided markets and the debate concerning net neutrality (e.g., Hagiu (2009) [64], Weyl (2010) [118], Economides and Tåg (2012) [42] and Brito, Pereira and Vareda (2013) [20]).

Hagiu (2009) [64] studies a market where indirect network effects are determined endogenously, through consumers' taste for variety and producer competition. He concludes that the optimal platform pricing structure shifts towards extracting more rents from producers relative to consumers when consumers have stronger demand for variety, since producers become less substitutable. The argument relies on the fact that, the consumer preferences for variety, the producer market power and the producer economies of scale in multihoming, make platforms' price-cutting strategies on the consumers' side less effective.

Weyl (2010) [118] develops a general theory of monopoly pricing of networks, where platforms use insulating tariffs to avoid coordination failure.

Economides and Tåg (2012) [42] discuss net neutrality in the context of a two-sided market. They find that under a monopoly platform, net neutrality regulation (requiring zero fees to content providers) may increase the total industry surplus as compared to the fully private optimum at which the monopoly platform imposes positive fees on content providers. Imposing net neutrality in duopoly with multihoming content providers and singlehoming consumers increases the total surplus as compared to duopoly competition with positive fees to content providers.

Brito, Pereira and Vareda (2013) [20] analyze the impact of network neutrality regulation on the competition between content providers (CP) and on the internet service providers incentives (ISPs) to invest. In their own words, "*if ISPs can offer network services of different quality to CPs, they prefer to sell the highest quality network services to the CP that collects the highest advertising revenues*". The authors also conclude that the impact of network neutrality regulation on the investment in the quality of network services is potentially ambiguous and depends on the symmetry between ISPs, and on the ISPs' ability to assign network's capacity to CPs. The authors conclude that "*if the ISPs are symmetric and have full discretion on how to allocate the level of quality of network services among CPs, investment and welfare are higher under the discriminatory regime*".

In the context of quality investments on platforms, Belleflamme and Peitz (2010) [14] show that for-profit intermediation may lead to overinvestment and free access leads to underinvestment since investment decisions influence the strength of indirect network effects and access prices.

Two-sided markets researchers also address special attention to the new phenomenon of multihoming and its relationship with compatibility between platforms (e.g., Doganoglu and Wright (2006) [38] and Armstrong and Wright (2007) [10]). Doganoglu and Wright (2006) [38] find that policymakers should be more worried about the lack of compatibility in the presence of multihoming and Armstrong and Wright (2007) [10] reveal the prevalence of exclusive contracts to prevent multihoming.

Recently, new studies address concerns on asymmetric network externalities (e.g., Ambrus and Argenziano (2009) [3]) and imperfect competition environments[9] (e.g., White and Weyl (2010) [119]). Incomplete information between platforms and agents and its relationship with agents expectations (myopic or forward looking) is also a relevant matter under the scrutiny of researchers (e.g., Gabszewicz and Wauthy (2004) [55], Yehezkel and Halaburda (2012) [123], Halaburda and Yehezkel (2013) [67] and Hagiu and Halaburda (2013) [65]).

Gabzsewicz and Wauthy (2004) [55] model duopoly competition between two platforms where the network effects are captured within a vertical differentiation framework. Under single-homing there exists an interior equilibrium where networks exhibit asymmetric sizes and both firms enjoy positive profits. When all agents are allowed to patronize the two platforms, they show that in equilibrium multi-homing takes place on one side of the market only.

Yehezkel and Halaburda (2012) [123] and Halaburda and Yehezkel (2013) [67] find that

[9] Kreps and Wilson (1982) [77] is a seminal contribution to an introduction on the literature embracing such environments.

under a competitive environment the incumbent dominates the market by setting the welfare-maximizing quantity when the difference in the degree of asymmetric information between buyers and sellers is sufficiently high. However and since a market failure may arise with asymmetric information, the authors state that multihoming solves this issue.

Hagiu and Halaburda (2013) [65] also perform an extensive study on the effect of different levels of information on two-sided platform profit, both under monopoly and duopoly competition to show that platforms "*with more market power (monopoly) prefer facing more informed users*". However, platforms with less market power get higher profits when users are less informed. They point out that the reason behind is "*that price information leads user expectations to be more responsive and therefore amplifies the effect of price reductions*".

The analysis of regulatory regimes in two-sided markets is also gaining importance (e.g., Evans (2003) [46], Wright (2004) [121], Rysman (2009) [98] and Valverde, Chakravorti and Fernandez (2009) [110]).

Evans (2003) [46] confirms that the antitrust analysis of two-sided markets should heed the economic principles that govern pricing and investment decisions. To shed some light on the impact of regulations on consumers, business and government, Wright (2004) [121] considers eight basic fallacies that can arise from using conventional wisdom from one-sided markets in two-sided market settings and discusses how these fallacies may be reconciled by proper use of a two-sided market analysis, based on empirical evidence.

Rysman (2009) [98] clarifies the understanding towards two-sided markets and alerts about the relevance of the literature on policy implications for pricing analysis. Valverde, Chakravorti and Fernandez (2009) [110] evaluate the impact of government intervention in these markets. Using confidential bank-level data to study the impact of government-encouraged fee reductions for payment card services, find that consumer and merchant welfare improved when the interchange fees and transfers among banks are reduced.

Besides theoretical developments, the empirical literature on two-sided markets has been focused on the estimation of the platforms' market power (e.g., Argentesi and Filistrucchi (2007) [6] and Song (2011) [107]), on the prediction of the effects of platforms' mergers (e.g., Chandra and Collard-Wexler (2009) [24], Filistrucchi, Klein and Michielsen (2012) [51] and Jeziorski (2012) [71]), on the analysis of the pricing structure (e.g., Kaiser and Wright (2006)

[72]), and forecasting platforms' demand (e.g., Lee (2011) [81]).

The literature also covers other traditional Industrial Organization topics such as entry (e.g., Zhu and Iansiti (2012) [124], Dewenter and Roesch (2012) [34]) and market dynamics (e.g., Vogelsang (2010) [115]).

Zhu and Iansiti (2012) [124] find that in the video games industry the entrant's success depends on the strength of the indirect inter-group externalities and on consumer's discount factor for future applications. Dewenter and Roesch (2012) [34] study the same matter to confirm that if the indirect network effects are small, the conventional results of market entry are applied, although weakened. However, for sufficiently strong inter-group externalities, a tipping structure can be optimal.

Vogelsang (2010) [115] presents a dynamic view of a two-sided market using Dixit's approach of entry deterrence to find that the platform has to solve the trade-off between the height of registration fees and marketing efforts in time: the platform optimally faces losses during the first years. Thus, points out that in social networks the leader does not exploit monopolistic profits at the beginning of its life cycle.

Despite the outstanding contributions already made available by the theoretical and empirical findings on two-sided markets, the fact is that this field is far from being exhausted and very challenging questions remain open. For example, one of the most recent developments in this literature arises from the recent controversy on Facebook survival (Cannarella and Spechler (2014) [23]).

In parallel, the recent ascent of the theory of two-sided markets has also stimulated the development of certain applied fields (namely, industries characterized by the presence of inter-group externalities), which benefit from the analytical frameworks developed within this theory. We highlight the recent advances on media economics (e.g., Anderson and Coate (2005) [5], Wilbur (2008) [120], Kaiser and Song (2009) [73], Dewenter, Haucap and Wenzel (2011) [33], Kind and Stähler (2011) [75], Yao and Mela (2011) [116], Albuquerque et al. (2012) [2], Reisinger (2012) [92] and Dietl, Lang, and Lin (2013) [37]), on telecommunications (e.g., Brito, Pereira and Vareda (2010) [18], Samanta, Woods and Ghanbari (2010) [99], Brito, Pereira and Vareda (2011) [19], Genakos and Valletti (2011) [60], Genakos and Valletti (2012) [61], Gans (2012) [59]), on financial markets (e.g., Rochet and Wright (2010) [96], Wang

(2010) [117], Bourreau and Verdier (2010) [16] and Verdier (2011) [114]), on healthcare issues (e.g., Bardey and Rochet (2010) [11]) and on sport events (e.g., Dietl et al. (2012) [36]), among others. Even considered an independent literature, the matching theory also adopts some features of two-sided markets (e.g., Kojima and Pathak (2009) [76], Halaburda (2010) [66] and Ackermann et al. (2011) [1].

Thus, the theory of two-sided markets is an emerging field under the scrutiny of researchers. To standardize the relevant features of the field, we now provide a quantitative analysis of the twelve-year history of this research line.

2.3 QUANTITATIVE ANALYSIS

We propose a categorization of a large dataset on the literature gathered from Scopus from 2002 to the end of 2013. Our analysis is an effort to uncover the research paths that brought two sided-markets into the microeconomics mainstream and so it is important to understand in which years more articles were released, who were the most relevant authors on the matter and which continent is leading the research.

All this information is relevant if someone intends to lead a future investigation on this field. We account for the relative frequency of articles published each year and we establish a comparison with the overall released articles in the field of Industrial Organization. Then, we also analyze the territory supremacy where two-sided markets authors are affiliated. Finally, we also present which economic journals support this line of research.

2.3.1 Methodology

We perform an analysis[10] of the field recurring to a database extracted from Scopus. The database was constructed using one search keyword: "two-sided markets". The search pro-

[10] Our focus relies only on the quantitative analysis of the field. Thus, we neglect any qualitative theoretical categorization of articles, which we intend to develop as a future research. To obtain a discussion on this topic check and follow Nelson and Winter (2009) [86] as well as the contributions of Yadav (2010) [122] and Silva and Teixeira (2011) [106], among others.

cedure covers the keyword "two-sided markets" in several dimensions: the title, the abstract and the keywords.

Although we recognize the limitations behind bibliometric exercises concerning the choice of the search keywords[11], we consider that the selected keyword is able to capture the core of the contributions under scrutiny. Since we want to focus only on research contributions, we neglect articles corresponding to comments, rejoinders, books, corrigendas or similar.

2.3.2 Scopus

Scopus is considered today one of the best tools for bibliometric exercises and evaluations of scientific production (Pato and Teixeira (2013) [90]) because it has the advantage to perform an immediate numerical analysis. Scopus also encompasses more modern sources (Chappin and Ligtvoet (2014) [25]), however, Scopus purportedly has a more European focus (Chappin and Ligtvoet (2014) [25]).

On a first (data treatment) stage, we use the export function to obtain all the relevant articles into an csv file. We combine the resulting csv files and generate a file with all authors' name, the title of each article, the year of publication, title of the journal, continent prominence and source.

In a second (qualitative) stage, in order to identify and evaluate the quality of the journals where two-sided markets articles are being published, we study the impact factor of the relevant international journals, often considered a proxy for the journals' scientific quality (Vieira and Teixeira (2010) [113])[12].

In a third (quantitative) stage, we perform the quantitative analysis, based on simple and exploratory statistics. We complete the search in the Scopus bibliographic database on the 15^{th}. of December, 2013. From the first stage procedure resulted a database of 371 published articles, involving 241 authors.

[11] See Santos and Teixeira (2009) [102] for a brief discussion on this topic. Also note that Scopus only allows for the extraction of scientific articles.

[12] The full information on this topic can be found on the journals' web site, in Scopus and in the Web of Knowledge.

2.3.3 The evolution of the research

Analyzing the number of articles on two-sided markets through time, we observe an increasing number of articles published since 2002 (cf. figure 1).

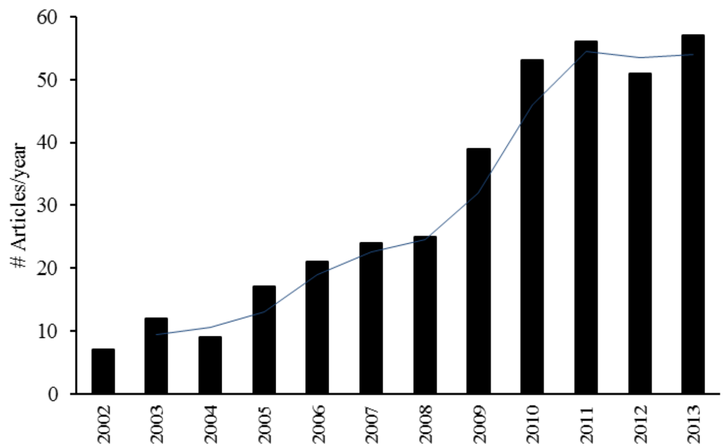

Figure 1: Number of articles published by year

We neglect articles before 2002 since there are no prominent investigations in this field and the number of manuscripts was rather negligible. From 2002, we observe that the number of two-sided markets articles increased gradually and, more specifically, it started to grow faster in the period 2008-2010, stabilizing after 2011. This path may be explained by three main causes: the widely spread concern with the antitrust cases against international credit card networks (Visa and MasterCard) in the United States, Europe and Australia around 2000 (Roson (2005) [97]), the growth of online markets and the discussion on the regulation of this new emergent markets. In this context, the analysis of two-sided markets and the corresponding impact in terms of welfare is catching more and more the attention of an increasing number of researchers.

Although the field of two-sided markets has an endless number of applications (e-commerce, urban and public economics, mathematics, matching theory, among others), it is viewed as an Industrial Organization subfield[13]. Comparing the total number of Industrial Organization articles and two-sided markets articles, we verify that the study of two-sided markets is growing within the literature of Industrial Organization (although representing, on average, around 1,5% of the research conducted in the field in the period considered). The weight of the two-sided markets research on Industrial Organization is increasing, especially since 2008 (cf. table 2).

# of Articles	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**
Two-sided markets	7	12	9	17	21	24
Adjusted two-sided markets	5	8	5	9	12	13
Industrial organization	778	1043	1104	1140	1269	1075
Relative frequency (%)	0.64	0.76	0.45	0.04	0.95	1.2
# of Articles	**2008**	**2009**	**2010**	**2011**	**2012**	**2013**
Two-sided markets	25	39	53	56	51	57
Adjusted two-sided markets	13	16	21	23	22	28
Industrial organization	1046	932	906	928	986	903
Relative frequency (%)	1.24	1.72	2.32	2.48	2.23	3.1

Table 2: Relative frequency of articles on two-sided markets on Industrial Organization

2.3.4 Territory supremacy

Europe and (North) America are leading continents in two-sided markets research[14]. Although the first manuscripts appear simultaneously in Europe and (North) America, the "spreading effect" got slightly higher expression on the European continent. Nevertheless,

[13] This evidence is due to the fact that in nearly 47% of the downloaded articles, we acknowledge the existence of a positive correlation between the keyword "two-sided markets" and the associated JEL code L (Industrial Organization). Aggregating the JEL code L with the JEL code D (Microeconomics), the percentage increases to 80%. The item "adjusted two-sided markets articles" (cf. table 2) corresponds to articles with JEL code L only.

[14] To obtain the results by continent we establish a link between the authors' affiliation and their correspondent institution location. This implies that, for instance, if an author has a Portuguese nationality but is affiliated to a Japanese research institution/university, it will be considered an article produced in Asia.

their supremacy is shared with (North) America since both together correspond to 80% of total publications (cf. figure 2).

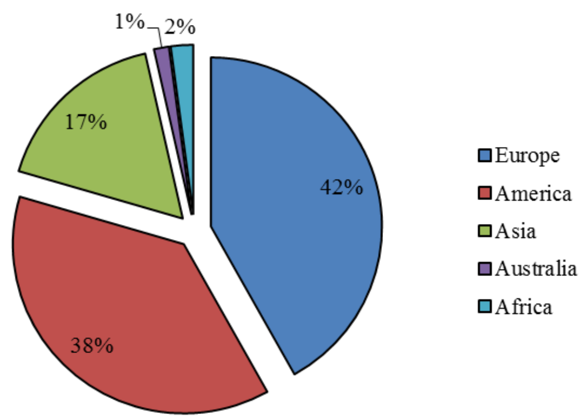

Figure 2: Number of articles published by continent

2.3.5 Main outlets and scientific notability

The great majority of journals that publish articles on two-sided markets are specialized in Industrial Organization. In addition, the topic has been covered by other specialized journals in Mathematics, Marketing and Finance. However, not only specialized journals are interested in two-sided markets, since more general journals have also published research in this line[15].

The outlets with the highest number of articles published on two-sided markets are the *International Journal of Industrial Organization* (with 16 articles), *Games and Economic Behavior* (with 12 articles) and *Information Economics and Policy* (with 9 articles).

Concerning the journals' impact factor (IF)[16], we find that the majority (around 70%) of the articles are published in journals with an impact factor higher than 1. If the interval

[15] For instance, Economics Letters and European Economic Review, among others have publications in this research field.

[16] The impact factor of an academic journal is a measure reflecting the average number of citations to recent articles published in the journal. Thus, it is frequently used as a proxy for the relative quality of a journal within its field of research, with journals with higher impact factors being considered more important than those with lower IF's.

is slightly relaxed, at least 85% are published in journals with an impact higher than 0.9, meaning that a significant number of publications in the field have at least moderate impact. Therefore, an increasing number of two-sided markets articles in combination with a strong perceived scientific quality of the field, are important explanations for the increasing prominence of two-sided markets[17] (cf. table 3). Also, it turns out that the majority of the economic journals have published at least one or two two-sided markets articles (cf. figure 3).

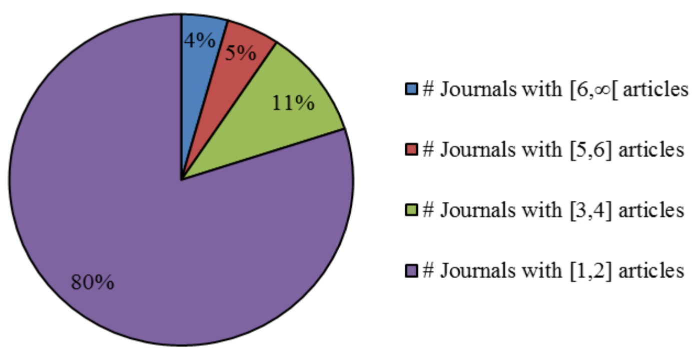

Figure 3: Relative frequency of publications in economic journals

[17]The journals considered in table 3 represent 50% of the total articles published in two-sided markets. In table 3, the label "Similar (a)" includes Management and Accounting and the label "Similar (b)" includes Econometrics and Finance. The impact factor values are from 2012. Source-Normalized Impact per Paper (SNIP) measures contextual citation impact by weighting citations based on the total number of citations in a subject field. It is defined as "the ratio of a journal's citation count per paper and the citation potential in its subject field". This is the measure we consider to find that around 70% of two-sided markets articles are published in journals with an IF higher than 1. SCImago Journal Rank (SJR) is a measure of scientific influence of scholarly journals that accounts for both the number of citations received by a journal and the importance or prestige of the journals where such citations come from.

Table 3: International economic journals, impact factor and subject area

Journal	Subject area	# nr.	SJR (SNIP)
International Journal of Industrial Organization	Business & Similar (a)	16	1.568 (1.478)
Games and Economic Behavior	Economics & Similar (b)	12	2.325 (1.742)
Information Economics and Policy	Economics & Similar (b)	9	0.616 (1.409)
Economics Letters	Economics & Similar (b)	8	0.807 (0.890)
Marketing Science	Business & Similar (a)	8	3.552 (1.786)
Review of Network Economics	Economics & Similar (b)	7	0.606 (1.282)
Journal of Banking and Finance	Economics & Similar (b)	7	1.348 (1.826)
International Journal of Game Theory	Decision Sciences	6	0.623 (1.079)
Journal of Competition Law and Economics	Economics & Similar (b)	6	0.556 (1.239)
Journal of Management Information Systems	Business & Similar (a)	5	1.598 (1.567)
Journal of Industrial Economics	Business & Similar (a)	5	2.028 (1.599)
Review of Industrial Organization	Business & Similar (a)	5	0.311 (0.874)
International Game Theory Review	Business & Similar (a)	5	0.239 (0.515)
Management Science	Business & Similar (a)	5	2.902 (2.223)
Info	Social Sciences	5	0.374 (0.536)
Economics Bulletin	Economics & Similar (b)	5	0.181 (0.255)
American Economic Review	Economics & Similar (b)	5	6.459 (3.380)
Economic Theory	Economics & Similar (b)	4	1.857 (1.364)
Xitong G. L. Y. S. System Engineering	Computer Science	4	0.279 (0.802)
Competition Policy International	Economics & Similar (b)	4	0.363 (0.698)
Journal of Economics and Management Strategy	Business & Similar (a)	4	1.496 (1.601)
Theory and Decision	Arts and Humanities	4	0.685 (0.963)
Journal of Economic Theory	Economics & Similar (b)	4	3.950 (1.739)
Telecommunications Policy	Computer Science	3	0.711 (1.681)
European Economic Review	Economics & Similar (b)	3	1.880 (1.733)
Journal of the European Economic Association	Economics & Similar (b)	3	4.160 (2.287)
International Economic Review	Economics & Similar (b)	3	3.541 (2.000)
Mathematics of Operations Research	Computer Science	3	1.846 (1.629)
American Economic Journal Microeconomics	Economics & Similar (b)	3	3.798 (1.683)

2.3.6 Main contributors

We now identify the most relevant authors within the field (cf. table 4)[18]. Most of the Top-10 prominent authors belong to universities from Europe and (North) America. These authors have a remarkable number of publications indexed in Scopus, a large number of citations and a high h-index[19], which reflects their scientific notability.

Ranking (author)	**Affiliation**	**# 2SM articles (Citations) [h index]**	**# total articles (Citations) [h index]**
1 (J. C. Rochet)	Universität Zurich	5 (555) [4]	66 (2565) [20]
2 (J. Tirole)	Toulouse School of Economics	3 (538) [3]	128 (9115) [40]
3 (M. Armstrong)	University of Oxford	2 (335) [2]	40 (1380) [16]
4 (M. Van Alstyne)	MIT Center for Digital Business	4 (314) [4]	28 (931) [10]
5 (J. Wright)	National University of Singapore	8 (226) [7]	34 (751) [15]
6 (N. Economides)	New York University	2 (116) [2]	29 (1124) [10]
7 (U. Kaiser)	Inst. for Study of Labor, Bönn	3 (65) [3]	23 (400) [9]
8 (M. Peitz)	Universität Mannheim	3 (43) [3]	53 (398) [11]
9 (B. Jullien)	Toulouse School of Economics	3 (23) [3]	36 (846) [12]
10 (A. Hagiu)	Harvard Business School	2 (22) [2]	11 (140) [6]

Table 4: Main contributors in two-sided markets

Taking into consideration three criteria: (*i*) the number of two-sided markets articles developed, (*ii*) the number of articles citations and (*iii*) the h-index of each author, the

[18] To complete an authors' ranking, we consider only the authors focused on IO literature, which is the scope of the thesis. Thus, we do not consider authors whose scopes of publication belong to independent literatures (e.g., F. Kojima in matching theory, among others). The quantitative approach reflects the application of standard bibliometric techniques.

[19] The h-index is an index that attempts to measure both the productivity of the published work of a research. The index is based on the set of the scientist's most cited papers and the number of citations that they have received in other publications. It is importance to mention that the author with more articles in this research field does not coincide with the one with more citations on articles of two-sided markets. Thus, the author prominence depends not only on the number of publications but also their closed relation with citations and the correspondent h-index.

ranking shows that the top-3 prominent authors are Jean Charles Rochet, Jean Tirole and Mark Armstrong.

Finally, the majority of researchers (90% according to Scopus) have developed one or two articles (cf. figure 4).

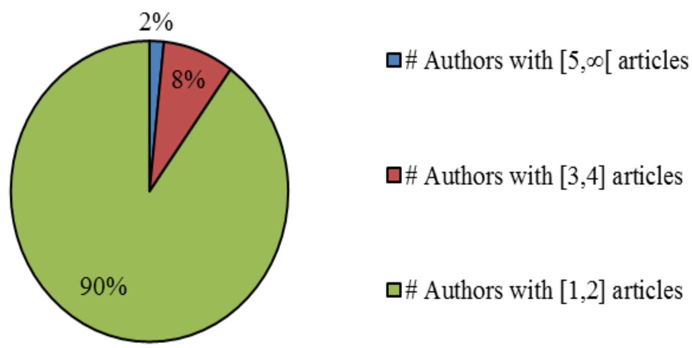

Figure 4: Number of articles published by author

In order to summarize, we remark the findings in the following conclusion.

Conclusion 1. ***(Stylized Facts on Two-Sided Markets)***

(*i*) *Two-sided markets published articles are increasing since 2002*;

(*ii*) *Most of two-sided markets articles are developed in Europe and (North) America*;

(*iii*) *There exists a high level of receptivity by international economic journals towards two-sided markets*;

(*iv*) *The prominent authors in two-sided markets are Jean Charles Rochet, Jean Tirole and Mark Armstrong*;

(*v*) *From the universe of two-sided markets researchers, the majority elaborates one or two manuscripts*;

2.4 CONCLUSIONS

In this chapter, we define and characterize the field of two-sided markets. We provide a literature review and we cover the recent developments on the field under scrutiny. We also propose a methodology to quantify the research on two-sided markets.

We conclude that two-sided markets published articles are increasing since 2002, the majority of two-sided markets articles have been published after 2008 and most of two-sided markets articles are developed in Europe and (north) America. We observe the existence of high levels of receptivity by international economic journals on two-sided markets and the prominent authors in two-sided markets are Jean Charles Rochet, Jean Tirole and Mark Armstrong.

This analysis is preliminary. We leave a relevant improvement for future research: the qualitative categorization of the articles in order to understand, for instance, if the majority of the articles embrace price (or quantity) competition, if the majority of manuscripts adopt singlehoming (or partial/total) multihoming agents, which fraction of articles conducts empirical validation and which fraction of articles are theoretical models and so on.

3.0 NESTING VERTICAL AND HORIZONTAL DIFFERENTIATION IN TWO-SIDED MARKETS

Abstract. We develop a model that is a synthesis of the two-sided markets duopoly model of Armstrong (2006) [9] with the nested vertical and horizontal differentiation model of Gabszewicz and Wauthy (2012) [57], which consists of a linear city with different consumer densities on the left and on the right side of the city. In equilibrium, the high-quality platform sells at a higher price and captures a greater market share than the low-quality platform, despite the indifferent consumer being closer to the high-quality platform. The difference between market shares is lower than socially optimal, because of the inter-group externality and because the high-quality platform sells at a higher price. We conclude that a perturbation that introduces a negligible difference between the consumer density on the left and on the right side of the city may disrupt the existence of equilibrium in the model of Armstrong (2006) [9]. Finally, we show that inter-group externalities make it easier to deter an inferior-quality entrant and make it easier for a superior-quality entrant to conquer the market.

Keywords: Two-sided markets, Horizontal differentiation, Vertical differentiation.

JEL Classification Numbers: D42, D43, L13.

3.1 INTRODUCTION

A two-sided market is a market where firms are platforms that allow two distinct groups of consumers to interact in order to engage in an activity that would not be possible without a platform. Examples in the literature include clubs, social networks and dating sites (men and women), online trading sites (buyers and sellers), newspapers (consumers and advertisers), video games (software developers and end users), yellow pages and credit cards (consumers and merchants), and others.

For example, in the clubs market, we can conceive that men and women constitute the two sides of the market. A given man chooses a certain club (platform) taking into consideration, among other aspects: the inter-group externality (number of women attending each of the clubs), the type of music (rock or latin), the environment (air conditioned, cleanliness), the access price and the travel time. There are, evidently, elements of both vertical and horizontal differentiation involved.

We study price competition between two platforms that sell horizontally and vertically differentiated products (whose quality is perfectly observed by both sides of the market) by developing a model that synthesizes the two-sided market model of Armstrong (2006) [9] and the product differentiation model of Gabszewicz and Wauthy (2012) [57].[1]

In the model of Gabszewicz and Wauthy (2012) [57], the competing platforms are located at opposite extremes of a linear city. Consumers, who are distributed along the city, face linear transportation costs to travel to the platform of their choice. The difference with respect to the standard model of horizontal differentiation (Hotelling (1929) [68]) is the fact that consumer density is different on the left side and on the right side of the city. This asymmetry of consumer density introduces vertical differentiation in the model.[2] To

[1]In most of the literature, competition with differentiated products is studied alternatively using models of horizontal or vertical differentiation. However, there are some contributions where horizontal and vertical differentiation are simultaneously considered, as Neven and Thisse (1990) [87], Economides (1989) [39], Irmen and Thisse (1998) [69], Daughety and Reinganum (2007) [29], Argenziano (2008) [7], Levin, Peck and Ye (2009) [82], Griva and Vettas (2011) [63], Gabszewicz and Wauthy (2012) [57] and Amir, Gabszewicz and Resende (2013) [4].

[2]Gabszewicz and Wauthy [57] define the natural market of a firm as the group of consumers who, at equal prices, prefer the variant offered by this firm relatively to the variant offered by its competitor. In the light of this, they generalize the model of Hotelling [68] to allow for natural

this environment, the two-sided market elements of the model of Armstrong (2006) [9] are added. There are two groups of consumers, which have the same distribution along the city and have the same benefit of interacting with agents of the other group.[3]

Considering weak network effects, we provide necessary and sufficient conditions for existence and uniqueness of an interior equilibrium and explain how prices, market shares and profits depend on the strength of inter-group externalities and on the degree of vertical differentiation. The equilibrium is unique and characterized by the fact that the high-quality platform charges a higher price and has a greater market share than the low-quality platform, although the indifferent consumer is located on the right side of the city centre.

The effect of the inter-group externality on prices is the same as in the model of Armstrong (2006) [9]: prices decrease in a way that exactly reflects the benefit of attracting agents on one side to agents on the other side. A stronger inter-group externality increases the market share of the high-quality platform, which is the more populated one, but not sufficiently to offset the negative impact of lower prices on its profit.

Vertical differentiation as defined by Gabszewicz and Wauthy (2012) [57] implies that the low-quality platform competes more fiercely, leading to lower prices at both platforms. In spite of facing a more fierce competition, the high-quality platform sees its market share increase. However, its profit decreases unless the inter-group externality is sufficiently strong.

We also study the existence of tipping equilibria, only to confirm that the more complicated consumer distribution of the model of Gabszewicz and Wauthy (2012) [57] does not interfere with the conditions for existence nor with the characteristics of tipping equilibria, which are exactly as in the model of Armstrong (2006) [9].

Considering the problem of a benevolent planner that allocates consumers to platforms in order to promote social welfare efficiency, we find that it is more efficient to have two active platforms than only one (under the assumption of weak network effects). Moreover,

markets of different sizes. In their own words: "*In the symmetric linear model with firms located at the extremities of the unit interval, natural markets are defined by the* $\left[0, \frac{1}{2}\right]$ *and* $\left[\frac{1}{2}, 1\right]$ *intervals, respectively. In order to allow for natural markets with different sizes, we then assume that the density differs from one interval to the other. Notice that in this model, a vertical configuration appears as a limiting case where the density of one of the intervals tends to zero while the density of the other tends to 1.*"

[3] Armstrong [9] does not restrict inter-group externalities to be symmetric in this sense.

we show that, in the market equilibrium, there are too few consumers on the high-quality platform. This is due to the inter-group externality (which is not taken into account by individual agents) and reinforced by the fact that the low-quality platform charges lower prices.[4]

Finally, we investigate two particular cases of possible entry as in Gabszewicz and Wauthy (2012) [57]. Our conclusions are that, in the presence of cross-network externalities, the deterrence of an inferior-quality entrant is easier and the conquest of the market by a superior-quality entrant can be achieved with a higher limit price.

The paper is organized as follows. Section 3.2 presents a description of the model. Section 3.2.1 provides the corresponding analysis. Section 3.2.2 studies and defines the region where the (interior) equilibrium exists as well the tipping region and Section 3.3 presents the socially optimal outcome promoted by a benevolent planner. Finally, Section 3.4 covers entry and Section 3.5 concludes. The appendix contains the proofs of most propositions.

3.2 THE MODEL

Consider a two-sided market with two platforms, A and B, that are horizontally and vertically differentiated. The platforms are exogenously located at opposite extremes of a linear city: platform A is located at $x = 0$ while platform B is located at $x = 1$. There are no production costs.

There is a continuum of consumers on each of the two sides of the market, 1 and 2. Consumers inelastically demand one unit of the service that is provided by the platform. We follow Armstrong (2006) [9] in considering linear inter-group externalities and Serfes and Zacharias (2012) [104] in assuming symmetry between the two sides of the market.

Horizontal differentiation is captured by transportation costs that are linear in distance, as in the model of Hotelling (1929) [68].

For technical convenience, the transportation cost parameter is set to 1, following Neven

[4] This is also the conclusion of Argenziano [7].

and Thisse (1990) [87] and Gabszewicz and Wauthy (2012) [57], among others.

As in Gabszewicz and Wauthy (2012) [57], vertical differentiation is captured by the fact that there are more consumers on the right half of the city than on the left half. On each side of the market, the density of consumers is μ at any $x \in \left[0, \frac{1}{2}\right]$ and $1 - \mu$ at any $x \in \left(\frac{1}{2}, 1\right]$.

Without loss of generality, we assume that $\mu \leq \frac{1}{2}$. Pure horizontal differentiation corresponds to the particular case in which $\mu = \frac{1}{2}$, while pure vertical differentiation corresponds to $\mu = 0$.

The two platforms, A and B, simultaneously choose their access prices, p^A and p^B, which apply to both sides of the market (price discrimination between sides is not allowed).

The utility of an agent of side j that is located at $x \in [0, 1]$ and chooses platform $i \in \{A, B\}$ is given by:

$$u_j^A(x) = V + \alpha D_k^A - p^A - x, \tag{3.1}$$

$$u_j^B(x) = V + \alpha D_k^B - p^B - (1 - x), \tag{3.2}$$

where V is the stand-alone benefit, α measures the strength of the inter-group externality and D_k^i is the number of consumers of side $k \neq j$ that join platform i.

Notice that, at equal prices and platform sizes, all consumers in $[0, \frac{1}{2})$ prefer to buy from platform A while all consumers in $(\frac{1}{2}, 1]$ prefer to buy from platform B. Since the density of consumers is greater at the right side of the line, this means that the majority of consumers prefers platform B. However, of course, agents also take into account the prices charged by the platforms and the platform choice by the agents of the other side.

The timing of the game is the following: in the first stage, platforms simultaneously set access prices for both sides. In the second stage, agents simultaneously choose which platform to join and their payoffs are determined.

3.2.1 Demand and profit functions

We assume that agents have rational expectations relatively to the platform choices of the other agents. This implies that all consumers to the left of the indifferent consumer choose platform A, while all consumers to the right choose platform B.

The hypothesis that platforms cannot price discriminate between the two-sides implies that the indifferent consumers of the two sides have the same location, i.e., that $\tilde{x}_1 = \tilde{x}_2$, allowing us to drop the subscripts and use the simpler notation p^A, p^B, $\tilde{x}$, D^A and D^B.

A consumer that is indifferent between platform A and B must be located at:[5]

$$\tilde{x} = \frac{1}{2}\left[1 + \alpha(D^A - D^B) - (p^A - p^B)\right]. \tag{3.3}$$

If $\tilde{x} < 0$, all consumers choose platform B; while if $\tilde{x} > 1$, everyone chooses platform A. Writing the demand for platform A from each consumer side as a function of the location of the indifferent consumer, we obtain:

$$D^A(\tilde{x}) = \begin{cases} \frac{1}{2}, & \text{if } \tilde{x} \geq 1; \\ (\tilde{x} - \frac{1}{2})(1-\mu) + \frac{\mu}{2}, & \text{if } \frac{1}{2} \leq \tilde{x} \leq 1; \\ \tilde{x}\mu, & \text{if } 0 \leq \tilde{x} \leq \frac{1}{2}; \\ 0, & \text{if } \tilde{x} \leq 0. \end{cases}$$

The following assumption, maintained throughout the paper except in Section 3.2.4, implies that the slope of the demand function is negative.

Assumption 3.1 (Weak inter-group externality) *The inter-group externality is relatively weak:* $\alpha < \frac{1}{1-\mu}$.

As a function of prices, the demand for platform A is given by (see Appendix 3.6.1):

$$D^A(p^A, p^B) = \begin{cases} \frac{1}{2}, & \text{if } p^A \leq p^B + \frac{\alpha}{2} - 1; \\ \frac{2(1-\mu)(p^B - p^A) + 2\mu - \alpha(1-\mu)}{4[1-\alpha(1-\mu)]}, & \text{if } p^B + \frac{\alpha}{2} - 1 \leq p^A \leq p^B - \frac{\alpha}{2}(1-2\mu); \\ \frac{2\mu(p^B - p^A) + \mu(2-\alpha)}{4(1-\alpha\mu)}, & \text{if } p^B - \frac{\alpha}{2}(1-2\mu) \leq p^A \leq p^B - \frac{\alpha}{2} + 1; \\ 0, & \text{if } p^A \geq p^B - \frac{\alpha}{2} + 1. \end{cases} \tag{3.4}$$

Since total demand is inelastic, the demand for platform B from each consumer side is $D^B(p^A, p^B) = \frac{1}{2} - D^A(p^A, p^B)$. The profit of each platform, $i \in \{A, B\}$, is given by $\pi^i(p^A, p^B) = 2p^i D^i(p^A, p^B)$.

[5] We sometimes refer to $\tilde{x}$ as the indifferent consumer with some abuse of language. More rigorously, it is the location (that may be outside the unit interval) where a consumer would be indifferent between the two platforms.

The demand and profit functions of platforms A and B are defined in four pieces, each corresponding to a different region where the indifferent consumer is located. A problematic characteristic of the function $D^A(\cdot, p^B)$ is the fact that its slope decreases from the second to the third subdomain in (3.4).[6]

This is illustrated in Figure $5(a)$, where it is assumed that $\alpha = 0.3$, $\mu = 0.1$ and $p^B = \frac{2-\mu}{3(1-\mu)} - \frac{\alpha}{2}$ (which, as we will see, is the equilibrium value of p^B for these values of the parameters α and μ). The profit function of platform A, $\pi^A(\cdot, p^B)$, is obtained by multiplying the demand function by $2p^A$. As illustrated in figure $6(a)$, it may have two local maxima.

Figure $5(b)$ illustrates the behavior of the demand of platform B, given the price set by platform A (again, we consider $\alpha = 0.3$, $\mu = 0.1$ and the equilibrium value of p^A). Since $D^B(p^A, \cdot)$ is concave, the profit function $\pi^B(p^A, \cdot)$ is also concave (cf. figure $6(b)$).

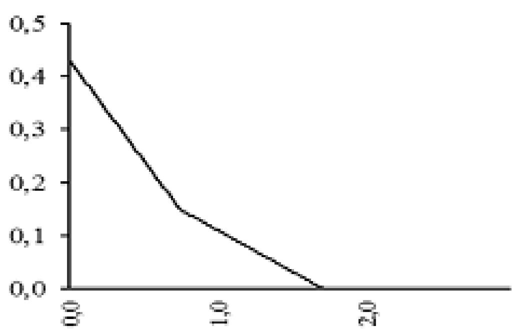

Figure 5(a) - D^A (y-axis) as a function of p^A (x-axis)

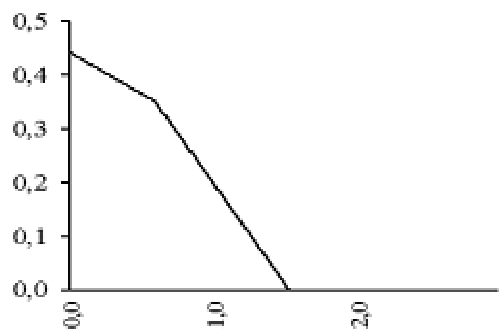

Figure 5(b) - D^B (y-axis) as a function of p^B (x-axis)

[6]This kink in the demand curve has appeared, for example, in the two-sided markets model of Resende (2010) [93], in models with "market inertia" or "switching costs" (Scotchmer, 1986 [103]; Farrell and Shapiro, 1986 [48]) and in a model with a particular kind of multihoming (Brandão et al., 2013 [17]).

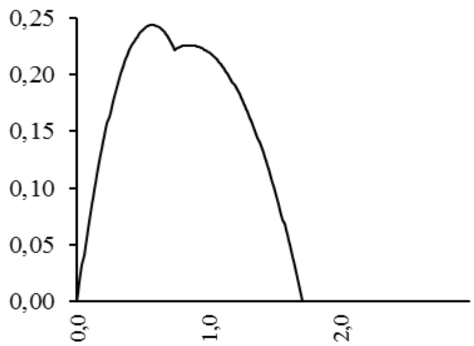

Figure 6(a) - π^A (y-axis) as a function of p^A (x-axis)

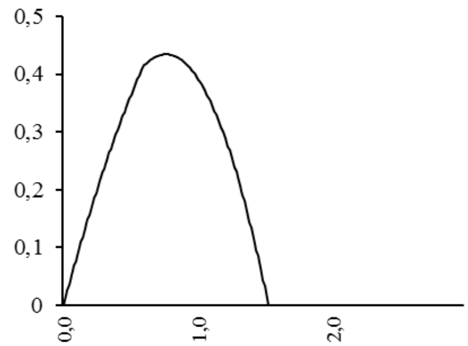

Figure 6(b) - π^B (y-axis) as a function of p^B (x-axis)

3.2.2 Interior equilibrium

We start by presenting the benchmark case of pure horizontal differentiation ($\mu = \frac{1}{2}$), which corresponds to the model of Armstrong (2006) [9] in the particular case of symmetric inter-group externalities.

Proposition 2. ***(Pure horizontal differentiation)***

If $\mu = \frac{1}{2}$, *there exists an interior equilibrium. It is such that:*

$$p^{A*} = p^{B*} = 1 - \tfrac{\alpha}{2},$$

$$\begin{cases} \tilde{x} = \frac{1}{2} \\ D^{A*} = D^{B*} = \frac{1}{4}, \end{cases}$$

$$\pi^{A*} = \pi^{B*} = \frac{1}{2} - \frac{\alpha}{4}.$$

Proof. See Appendix 3.6.2. □

In the presence of vertical differentiation, if the inter-group externality is not too strong, there exists an interior equilibrium with the indifferent consumer located on the right side of the city ($\tilde{x} > \frac{1}{2}$).

Proposition 3. ***(Horizontal and vertical differentiation)***

If $\mu < \frac{1}{2}$*, there exists an interior equilibrium if and only if* $\alpha \leq \frac{4-\mu}{6(1-\mu)} - \frac{\sqrt{12\mu+\mu^2}}{6(1-\mu)}$*. It is such that:*

$$\begin{cases} p^{A*} = \frac{1+\mu}{3(1-\mu)} - \frac{\alpha}{2} \\ p^{B*} = \frac{2-\mu}{3(1-\mu)} - \frac{\alpha}{2}, \end{cases}$$

$$\begin{cases} D^{A*} = \frac{2(1+\mu)-3\alpha(1-\mu)}{12[1-\alpha(1-\mu)]} \\ D^{B*} = \frac{2(2-\mu)-3\alpha(1-\mu)}{12[1-\alpha(1-\mu)]}, \end{cases}$$

$$\begin{cases} \pi^{A*} = \frac{[2(1+\mu)-3\alpha(1-\mu)]^2}{36(1-\mu)[1-\alpha(1-\mu)]}, \\ \pi^{B*} = \frac{[2(2-\mu)-3\alpha(1-\mu)]^2}{36(1-\mu)[1-\alpha(1-\mu)]}. \end{cases}$$

Proof. *See Appendix* 3.6.2. □

In the presence of vertical differentiation, the high-quality platform sets a higher price ($p^{B*} > p^{A*}$) and captures a greater market share ($D^{B*} > D^{A*}$) than the low-quality platform. Therefore, it also has higher profits ($\pi^{B*} > \pi^{A*}$).

We remark that, for the above equilibrium prices, the demand is unique, in the sense that there is no other equilibria of the game in which consumers, for given prices, choose which platform to join (see Appendix 3.6.3).

Observe that, when $\mu \to \frac{1}{2}$, the equilibrium variables converge to their values in the equilibrium with pure horizontal differentiation.

In the case of pure vertical differentiation ($\mu = 0$), which is the other extreme case, we obtain: $p^{A*} = \frac{1}{3} - \frac{\alpha}{2}$ and $p^{B*} = \frac{2}{3} - \frac{\alpha}{2}$; $D^{A*} = \frac{2-3\alpha}{12(1-\alpha)}$ and $D^{B*} = \frac{4-3\alpha}{12(1-\alpha)}$; $\pi^{A*} = \frac{1}{9(1-\alpha)} - \frac{\alpha(4-3\alpha)}{12(1-\alpha)}$ and $\pi^{B*} = \frac{1}{4(1-\alpha)} - \frac{\alpha(8-3\alpha)}{12(1-\alpha)}$.

As a corollary, we conclude that a perturbation that introduces a negligible difference between the consumer density on the left and on the right side of the city may disrupt the existence of equilibrium in the model of Armstrong (2006) [9].

Corollary 4. ***(No equilibrium)***

If $\mu < \frac{1}{2}$ *and* $\alpha > \frac{4-\mu}{6(1-\mu)} - \frac{\sqrt{12\mu+\mu^2}}{6(1-\mu)}$*, there exists no interior equilibrium.*

Proof. *This result follows from Proposition 3 and Lemmas 7 and 8 in Appendix 3.6.2.* □

3.2.3 Comparative statics

We now discuss the impact of the degree of vertical differentiation and of the strength of the inter-group externality on equilibrium prices, market shares and profits. All calculations are presented in Appendix 3.6.4.

As the degree of vertical differentiation increases (μ becomes lower)[7], both prices decrease, with the price of the low-quality platform decreasing faster ($\frac{\partial p^{A*}}{\partial \mu} > \frac{\partial p^{B*}}{\partial \mu} > 0$). Despite selling at an increasing discount relatively to the high-quality platform, the low-quality platform loses market share ($\frac{\partial D^{A*}}{\partial \mu} > 0$), and, therefore, also loses profits ($\frac{\partial \pi^{A*}}{\partial \mu} > 0$). Since the market share of the high-quality platform increases, the impact of vertical differentiation on its profit is not obvious. We find that the profit of the high-quality platform decreases if and only if the inter-group externality is weak ($\frac{\partial \pi^{B*}}{\partial \mu} > 0 \Leftrightarrow \alpha < \frac{2\mu}{1-\mu}$). It may be surprising that the profit of the high-quality platform may decrease when the degree of vertical differentiation increases. It follows from the fact that the price of the high-quality platform goes down as μ decreases due to an intensification of price competition.[8]

Prices decrease as the inter-group externality becomes stronger ($\frac{\partial p^{A*}}{\partial \alpha} = \frac{\partial p^{B*}}{\partial \alpha} = -\frac{1}{2}$), because attracting consumers on one side generates gains on the other side. More precisely, both prices are adjusted downwards by $\frac{\alpha}{2}$, which corresponds to the marginal benefit on the other side that results from attracting an additional agent on one side. The difference between the two prices does not depend on the strength of the inter-group externality. It is natural, therefore, that market shares diverge as the strength of the inter-group externality increases ($\frac{\partial D^{A*}}{\partial \alpha} < 0$).

Without inter-group externalities ($\alpha = 0$), we recover the setting of Gabszewicz and Wauthy (2012) [57] and, of course, their equilibrium prices ($p^{A*} = \frac{1+\mu}{3(1-\mu)}$ and $p^{B*} = \frac{2-\mu}{3(1-\mu)}$).

As the strength of the inter-group externality increases, the profit of platform A decreases, since price and market share strictly decrease. The profit of platform B also decreases, despite the increase in its market share.

[7] Recall that $\mu \in [0, \frac{1}{2}]$ is the consumer density on the left side of the city, while 1-μ is the consumer density on the right side of the city. Therefore, the lower is μ, the greater is the asymmetry between the sizes of the firms' natural markets.

[8] This effect is already present in Gabszewicz and Wauthy [57].

3.2.4 Tipping equilibria

Since the utility of accessing a platform is an increasing function of the platforms' membership, platform growth tends to be self-reinforcing and, therefore, one platform may capture the whole market (Katz and Shapiro, 1985 [74]). We conclude that vertical differentiation of the type that we are considering does not interfere with the condition for existence of tipping equilibria.

Proposition 5. ***(Tipping equilibrium)***

There exist tipping equilibria with $\tilde{x} = 0$ and $\tilde{x} = 1$ if and only if $\alpha > 2$.

Proof. See Appendix 3.6.2. □

3.3 SOCIAL OPTIMUM

Suppose that a benevolent planner can allocate consumers to the two platforms. Some questions arise. It is socially optimal to place all consumers in the high-quality platform, so that inter-group externalities are maximized? Or is it preferable to split consumers between the two platforms, to save on transportation costs?

Social welfare is given by:

$$W = \begin{cases} 2\int_0^{\tilde{x}} \left[V + \alpha D^A - x\right] \mu dx + 2\int_{\tilde{x}}^{\frac{1}{2}} \left[V + \alpha D^B - (1-x)\right] \mu dx \\ \quad +2\int_{\frac{1}{2}}^{1} \left[V + \alpha D^B - (1-x)\right] (1-\mu) dx, \quad \text{if } \ \tilde{x} \leq \dfrac{1}{2}; \\ 2\int_0^{\frac{1}{2}} (V + \alpha D^A - x)\mu dx + 2\int_{\frac{1}{2}}^{\tilde{x}} (V + \alpha D^A - x)(1-\mu) dx \\ \quad +2\int_{\tilde{x}}^{1} (V + \alpha D^B - (1-x))(1-\mu) dx, \quad \text{if } \ \tilde{x} > \dfrac{1}{2}. \end{cases}$$

Conditionally on $\tilde{x} \in \left[0, \frac{1}{2}\right]$, social welfare as a function of $\tilde{x}$ is given by:

$$W(\tilde{x}) = V - \frac{1}{4} + \frac{\alpha}{2} - \frac{\mu}{2} + \tilde{x}^2 \left[-2\mu(1 - 2\alpha\mu)\right] + \tilde{x}\left[2\mu(1-\alpha)\right]. \tag{3.5}$$

If $\alpha < 1$, the maximum in $\left[0, \frac{1}{2}\right]$ is attained at:

$$\tilde{x}^{FB} = \frac{1-\alpha}{2\left(1 - 2\alpha\mu\right)}. \tag{3.6}$$

This constrained maximizer is, in fact, the global maximizer.

Proposition 6. ***(Social optimum versus market equilibrium)***

Let $\alpha < 1$. In the socially optimal outcome, both platforms operate in the market, with the high-quality platform capturing a greater market share than in the market equilibrium. For $\alpha > 1$, the high-quality platform captures the whole market.

Proof. *See Appendix 3.6.5.* □

Comparing the socially optimal outcome with the market equilibrium outcome, we conclude that there are too many consumers on the low-quality platform at the market equilibrium.

3.4 ENTRY

The extension of the model to a triopoly is particularly relevant if we consider the possibility of entry by a third firm. In line with Gabszewicz and Wauthy [57], we study entry by an inferior-quality platform and entry by a superior-quality platform. We aim at understanding whether inter-group externalities make it easier or harder for the incumbents to deter the entry of an inferior-quality platform and for a superior-quality platform to capture the whole market.

3.4.1 Deterrence of an inferior-quality entrant

Suppose that a third platform, C, has the possibility of entering the market, becoming positioned at a distance L from the center of the city (i.e., from $x = \frac{1}{2}$), in a direction that is orthogonal to the linear city.

Any $L > \frac{1}{2}$ implies that no consumer would choose platform C if all platforms charged the same prices and had the same number of customers on the other side of the market.

As in Gabszewicz and Wauthy (2012) [57], suppose that platforms A and B are charging the pre-entry equilibrium prices established by Proposition **3**. Can platform C attract any consumers by charging $p^C = 0$, given that $D^C = 0$?[9]

It can if and only if it can attract the consumer located at $x = \frac{1}{2}$:

$$V - L \geq V + \alpha D^{A*} - p^{A*} - \frac{1}{2} \;\Leftrightarrow\; L \leq p^{A*} - \alpha D^{A*} + \frac{1}{2}$$
$$\Leftrightarrow\; L \leq \frac{1+\mu}{3(1-\mu)} + \frac{1}{2} - \frac{\alpha}{2} - \alpha\frac{2(1+\mu) - 3\alpha(1-\mu)}{12\left[1-\alpha(1-\mu)\right]}.$$

When $\alpha = 0$, we recover the result of Gabszewicz and Wauthy (2012) [57]. The presence of inter-group externalities diminishes the critical value of L for which entry is deterred, i.e., it helps platforms A and B to deter the entrance of platform C.

3.4.2 Market capture by a superior-quality entrant

Now suppose that the potential entrant offers a service of superior quality, in the sense that its stand-alone value is $\hat{V} > V$.

Assuming that $\hat{V} > V + L + \frac{1}{2}$ implies that all consumers would choose platform C if all platforms charged the same prices and had the same number of customers on the other side of the market.

Can platform C capture the whole market, even with platforms A and B charging a null price? The answer is positive if and only if:

$$\hat{V} + \frac{\alpha}{2} - L - \frac{1}{2} \geq V \;\Leftrightarrow\; L \leq \hat{V} - V + \frac{\alpha}{2} - \frac{1}{2}.$$

If the above condition is satisfied, prices $p^{A*} = p^{B*} = 0$ and $p^{C*} = \hat{V} - V + \frac{\alpha}{2} - \frac{1}{2} - L$, together with demand $D^{A*} = D^{B*} = 0$ and $D^{C*} = \frac{1}{2}$, constitute a triopoly equilibrium (unless platform C prefers to charge a higher price in spite of losing some demand).

Notice that the presence of inter-group externalities makes it easier for the superior quality entrant to capture the whole market.

[9] Notice that we are considering that the status quo is the equilibrium of the baseline model, i.e., that consumers are already in the incumbent platforms. The entrant faces, therefore, an additional difficulty in attracting consumers. Coordination is, in this sense, assumed to be adverse to the entrant.

3.5 CONCLUSIONS

We have studied competition between two horizontally and vertically differentiated platforms in a two-sided market, using a model that captures those of Armstrong (2006) [9] and Gabszewicz and Wauthy (2012) [57] as particular cases for pure horizontal differentiation and no inter-group externalities, respectively.

In spite of the technical difficulties that are intrinsic to the model of Gabszewicz and Wauthy (2012) [57], we have been able to fully characterize the existence and uniqueness properties of equilibrium under the assumption that the inter-group externality is relatively weak. One important aspect of our existence results is that a perturbation of the model of Armstrong (2006) [9] that introduces a small jump in the consumer density at the center of the city disrupts the existence of equilibrium.

The equilibrium properties essentially combine the features of the two-sided market model of Armstrong (2006) [9], in particular, the fact that prices are lower due to the benefit on the other side that results from increased membership on one side, with those of the nested horizontal and vertical differentiation model of Gabszewicz and Wauthy (2012) [57], where the low-quality firm becomes a more fierce competitor due to the decrease of the average consumer density coupled with the increase of the marginal consumer density.

Comparing the socially optimal outcome with the market equilibrium, we concluded that a benevolent planner would prefer to increase the market share of the high-quality platform. This was also the conclusion of Argenziano (2008) [7] in her study of a duopoly with product differentiation and network effects.

Finally, comparing our results with those of Gabszewicz and Wauthy (2012) [57] we concluded that inter-group externalities facilitate the deterrence of an inferior-quality entrant and the capture of the whole market by a superior-quality entrant.

3.6 APPENDIX

3.6.1 Demand

The four consumer utilities assuming the existence of platforms $i \in \{A, B\}$ and sides $j \in \{1, 2\}$, $k \in \{2, 1\}$ and $\alpha = \alpha_1 = \alpha_2$ and $\mu = \mu_1 = \mu_2$ are:

$$\begin{cases} u_1^A(x_1) = V + \alpha D_2^A - p_1^A - x_1; \\ u_1^B(x_1) = V + \alpha D_2^B - p_1^B - (1 - x_1); \\ u_2^A(x_2) = V + \alpha D_1^A - p_2^A - x_2; \\ u_2^B(x_2) = V + \alpha D_1^B - p_2^B - (1 - x_2). \end{cases}$$

The indifferent consumer on both sides of the market follows by setting: $u_1^A(x_1) = u_1^B(x_1)$ and $u_2^A(x_2) = u_2^B(x_2)$. Straightforward calculations imply:

$$\begin{cases} \tilde{x}_1 = \frac{1}{2} + \frac{\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B)}{2} \\ \tilde{x}_2 = \frac{1}{2} + \frac{\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B)}{2}. \end{cases}$$

3.6.1.1 Assuming that the indifferent consumers are on the right:

demands for $\tilde{x}_i \in \left[\frac{1}{2}, 1\right]$, $i \in \{1, 2\}$, are defined by:

$$\begin{cases} D_i^A(\tilde{x}_i) = \frac{\mu}{2} + (\tilde{x}_i - \frac{1}{2})(1 - \mu); \\ D_i^B(\tilde{x}_i) = (1 - \tilde{x}_i)(1 - \mu). \end{cases}$$

Substituting the expressions for $\tilde{x}_1$ and $\tilde{x}_2$, we obtain:

$$\begin{cases} D_1^A = \frac{\mu}{2} + \frac{1-\mu}{2}\left[\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B)\right]; \\ D_2^A = \frac{\mu}{2} + \frac{1-\mu}{2}\left[\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B)\right]; \\ D_1^B = \frac{1-\mu}{2}\left[1 - \alpha(D_2^A - D_2^B) + (p_1^A - p_1^B)\right]; \\ D_2^B = \frac{1-\mu}{2}\left[1 - \alpha(D_1^A - D_1^B) + (p_2^A - p_2^B)\right]. \end{cases}$$

Solving this linear system for $p^A = p_1^A = p_2^A$ and $p^B = p_1^B = p_2^B$, we obtain:

$$\begin{cases} D^A(p^A, p^B) = \frac{(1-\mu)(p^B - p^A)}{2[1-\alpha(1-\mu)]} + \frac{2\mu - \alpha(1-\mu)}{4[1-\alpha(1-\mu)]}; \\ D^B(p^A, p^B) = \frac{(1-\mu)(p^A - p^B)}{2[1-\alpha(1-\mu)]} + \frac{(2-\alpha)(1-\mu)}{4[1-\alpha(1-\mu)]}. \end{cases} \tag{3.7}$$

3.6.1.2 Assuming that the indifferent consumers are on the left: demands for $\tilde{x}_i \in \left[0, \frac{1}{2}\right]$, $i \in \{1,2\}$, are defined by:

$$\begin{cases} D_i^A(\tilde{x}_i) = \tilde{x}_i\mu; \\ D_i^B(\tilde{x}_i) = (\frac{1}{2} - x_1)\mu + \frac{1}{2}(1-\mu). \end{cases}$$

Once we substitute the respective demands by $\tilde{x}_1$ and $\tilde{x}_2$, we obtain:

$$\begin{cases} D_1^A = \frac{\mu}{2} + \frac{\mu}{2}\left[\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B)\right]; \\ D_2^A = \frac{\mu}{2} + \frac{\mu}{2}\left[\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B)\right]; \\ D_1^B = \frac{1-\mu}{2} - \frac{\mu}{2}\left[\alpha(D_2^A - D_2^B) - (p_1^A - p_1^B\right]; \\ D_2^B = \frac{1-\mu}{2} - \frac{\mu}{2}\left[\alpha(D_1^A - D_1^B) - (p_2^A - p_2^B\right]. \end{cases}$$

Solving this linear system for $p^A = p_1^A = p_2^A$ and $p^B = p_1^B = p_2^B$, we obtain:

$$\begin{cases} D^A(p^A, p^B) = \frac{2\mu(p^B - p^A)}{4(1-\alpha\mu)} + \frac{(2-\alpha)\mu}{4(1-\alpha\mu)}; \\ D^B(p^A, p^B) = \frac{2\mu(p^A - p^B)}{4(1-\alpha\mu)} + \frac{2-2\mu-\alpha\mu}{4(1-\alpha\mu)}. \end{cases} \tag{3.8}$$

3.6.2 Equilibrium prices, demand and profits

Proof of Proposition 2

In the particular case of $\mu = \frac{1}{2}$, demand and profits have the same analytical expression on the left and on the right side of the city. Thus, there is no longer a kink in the demand function.

From (3.7), the first-order conditions for profit maximization are:

$$\begin{cases} \frac{\partial \pi^A}{\partial p^A} = 0 \\ \frac{\partial \pi^B}{\partial p^B} = 0 \end{cases} \Leftrightarrow \begin{cases} 2\left(1+\frac{\alpha}{2}\right)p^A - \left(1+\frac{\alpha}{2}\right)p^B = \left(1-\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right) \\ 2\left(1+\frac{\alpha}{2}\right)p^B - \left(1+\frac{\alpha}{2}\right)p^A = \left(1-\frac{\alpha}{2}\right)\left(1+\frac{\alpha}{2}\right) \end{cases}$$
$$\Leftrightarrow\ p^A = p^B = 1 - \frac{\alpha}{2}.$$

The equilibrium prices satisfy the second-order condition, because:

$$\frac{\partial^2 \pi^A}{\partial {p^A}^2} = \frac{-\frac{1}{2}}{1 - \frac{\alpha^2}{4}},$$

which is strictly negative under Assumption 3.1. □

Lemma 7.

There exists no equilibrium with $\tilde{x} \in \left(0, \frac{1}{2}\right)$.

Proof. We will use the first-order conditions to find a single candidate equilibrium with the indifferent consumer located at the left of the city center, and, then, we will show that the candidate equilibrium cannot be an actual equilibrium

We will use the superscript "+" to denote the values of prices, demands and profits in that candidate equilibrium.

(i) First-order conditions for profit-maximization

Considering the demand functions for $\tilde{x}_i \in \left(0, \frac{1}{2}\right)$, $i \in \{1, 2\}$, which are given by (3.8), the first-order conditions for profit maximization yield:

$$\begin{cases} \frac{\partial \pi^A}{\partial p_1^A} = 0 \\ \frac{\partial \pi^A}{\partial p_2^A} = 0 \\ \frac{\partial \pi^B}{\partial p_1^B} = 0 \\ \frac{\partial \pi^B}{\partial p_2^B} = 0 \end{cases} \Leftrightarrow \begin{cases} 4\mu p_1^A + 4\alpha\mu^2 p_2^A - 2\mu p_1^B - 2\alpha\mu^2 p_2^B = -\alpha^2\mu^2 + 2\alpha\mu^2 - \alpha\mu + 2\mu; \\ 4\alpha\mu^2 p_1^A + 4\mu p_2^A - 2\alpha\mu^2 p_1^B - 2\mu p_2^B = -\alpha^2\mu^2 + 2\alpha\mu^2 - \alpha\mu + 2\mu; \\ -2\mu p_1^A - 2\alpha\mu^2 p_2^A + 4\mu p_1^B + 4\alpha\mu^2 p_2^B = -\alpha^2\mu^2 - 2\alpha\mu^2 + \alpha\mu - 2\mu + 2; \\ -2\alpha\mu^2 p_1^A - 2\mu p_2^A + 4\alpha\mu^2 p_1^B + 4\mu p_2^B = -\alpha^2\mu^2 - 2\alpha\mu^2 + \alpha\mu - 2\mu + 2. \end{cases}$$

Solving this system of equations, we obtain the following equilibrium prices:

$$\begin{cases} p^{A+} = \frac{1+\mu}{3\mu} - \frac{\alpha}{2}, \\ p^{B+} = \frac{2-\mu}{3\mu} - \frac{\alpha}{2}. \end{cases} \tag{3.9}$$

The corresponding market shares are obtained by substituting (3.9) into (3.4):

$$\begin{cases} D^{A+} = \frac{2(1+\mu)-3\alpha\mu}{12(1-\alpha\mu)}, \\ D^{B+} = \frac{2(2-\mu)-3\alpha\mu}{12(1-\alpha\mu)}. \end{cases} \tag{3.10}$$

Profits are obtained from (3.9) and (3.10):

$$\begin{cases} \pi^{A+} = \frac{[2(1+\mu)-3\alpha\mu]^2}{36\mu(1-\alpha\mu)}, \\ \pi^{B+} = \frac{[2(2-\mu)-3\alpha\mu]^2}{36\mu(1-\alpha\mu)}. \end{cases} \tag{3.11}$$

(ii) Interiority of the indifferent consumer

The indifferent consumer is located at:

$$\tilde{x}^{+} = \frac{1}{2} - \frac{\alpha(1-2\mu)}{12(1-\alpha\mu)} + \frac{1-2\mu}{6\mu}. \tag{3.12}$$

It must be on the left half of the city, otherwise the candidate equilibrium cannot be an actual equilibrium. This requires that the inter-group externality is relatively strong:

$$\tilde{x}^{+} < \frac{1}{2} \;\Leftrightarrow\; \frac{\alpha(1-2\mu)}{12(1-\alpha\mu)} > \frac{1-2\mu}{6\mu} \;\Leftrightarrow\; \alpha\mu > 2 - 2\alpha\mu \;\Leftrightarrow\; \alpha > \frac{2}{3\mu}. \tag{3.13}$$

The indifferent consumer must also be located inside the city. This requires that the inter-group externality is relatively weak:

$$\begin{aligned}
\tilde{x}^{+} > 0 \;&\Leftrightarrow\; \frac{1+\mu}{\mu} > \frac{\alpha(1-2\mu)}{2(1-\alpha\mu)} \\
&\Leftrightarrow\; 2(1+\mu-\alpha\mu-\alpha\mu^2) > \alpha\mu - 2\alpha\mu^2 \\
&\Leftrightarrow\; 2+2\mu > 3\alpha\mu \\
&\Leftrightarrow\; \alpha < \frac{2(1+\mu)}{3\mu}.
\end{aligned} \tag{3.14}$$

This is always true under Assumption 3.1. Combining (3.13) with Assumption 3.1, we obtain $\alpha \in \left(\frac{2}{3\mu}, \frac{1}{1-\mu}\right)$, which is non-empty if and only if $\mu > 0.4$.

(iii) Global profit-maximization

Even if the local second-order conditions are satisfied, since the profit function of platform A is not globally quasi-concave, we must check that the local maximum determined before (candidate equilibrium price p^{A+}) is a global maximum (given p^{B+}).

For this to be true, it is necessary that:

$$\pi^A\left(p^{A+}, p^{B+}\right) \geq \pi^A\left(p^A, p^{B+}\right), \; \forall p^A. \tag{3.15}$$

A possible deviation is to p^{AR}, which is the local maximum of $\pi^A\left(\cdot, p^{B+}\right)$ in the branch that leads to $\tilde{x} \in \left[\frac{1}{2}, 1\right]$. From (3.4), in this branch, the profit function is given by:

$$\pi^A\left(p^A, p^{B+}\right) = 2p^A \tfrac{1-\mu}{2[1-\alpha(1-\mu)]}\left(p^{B+} - p^A + \tfrac{\mu}{1-\mu} - \tfrac{\alpha}{2}\right). \tag{3.16}$$

In this branch, the first-order condition for profit maximization yields:

$$\begin{aligned}
&\tfrac{\partial \pi^A\left(p^A,p^{B+}\right)}{\partial p^A}=0\\
\Leftrightarrow\ &\tfrac{1-\mu}{2[1-\alpha(1-\mu)]}\left(p^{B+}-p^{AR}+\tfrac{\mu}{1-\mu}-\tfrac{\alpha}{2}\right)=p^{AR}\tfrac{1-\mu}{2[1-\alpha(1-\mu)]}\\
\Leftrightarrow\ &2p^{AR}=p^{B+}+\tfrac{\mu}{1-\mu}-\tfrac{\alpha}{2}.
\end{aligned}$$

Replacing in expression (3.16), we obtain:

$$\pi^A\left(p^{AR},p^{B+}\right)=\tfrac{1-\mu}{4[1-\alpha(1-\mu)]}\left(p^{B+}+\tfrac{\mu}{1-\mu}-\tfrac{\alpha}{2}\right)^2.$$

Using (3.9), we find:

$$\pi^A\left(p^{AR},p^{B+}\right)=\tfrac{\left[2-3\mu+4\mu^2-3\mu\alpha(1-\mu)\right]^2}{36\mu^2(1-\mu)[1-\alpha(1-\mu)]}.$$

The equilibrium profit is not lower than this deviation profit if and only if:

$$\begin{aligned}
&\tfrac{[2(1+\mu)-3\alpha\mu]^2}{36\mu(1-\alpha\mu)}\geq\tfrac{\left[2-3\mu+4\mu^2-3\mu\alpha(1-\mu)\right]^2}{36\mu^2(1-\mu)[1-\alpha(1-\mu)]}\\
\Leftrightarrow\quad &-9(1-\mu)\mu^2\alpha^2+3\mu\left(4-2\mu+\mu^2\right)\alpha-4-5\mu^2\geq 0.
\end{aligned}$$

The roots of this quadratic expression in α are:

$$\alpha_c=\frac{1}{2\mu\left(1-\mu\right)}+\frac{1-\mu}{6\mu}\mp\frac{\sqrt{-8+16\mu+\mu^2}}{6\left(1-\mu\right)}.$$

For $\mu<8-6\sqrt{2}\approx 0.485$, the roots are complex, which means that the inequality never holds. For $\mu\geq 8-6\sqrt{2}$, the inequality holds for α between the two real roots. In that case, a necessary condition for (3.15) to hold is:

$$\alpha\geq\frac{1}{2\mu\left(1-\mu\right)}+\frac{1-\mu}{6\mu}-\frac{\sqrt{-8+16\mu+\mu^2}}{6\left(1-\mu\right)}.$$

However, the above condition is incompatible with condition (3.14):

$$\begin{aligned}
&\frac{1}{2\mu\left(1-\mu\right)}+\frac{1-\mu}{6\mu}-\frac{\sqrt{-8+16\mu+\mu^2}}{6\left(1-\mu\right)}<\frac{2(1+\mu)}{3\mu}\\
\Leftrightarrow\ &3+1-2\mu+\mu^2-\mu\sqrt{-8+16\mu+\mu^2}-4(1-\mu^2)<0\\
\Leftrightarrow\ &-2+5\mu<\sqrt{-8+16\mu+\mu^2}.
\end{aligned}$$

Since $\mu>0.4$, this condition is equivalent to: $2\mu^2-3\mu+1<0\ \Leftrightarrow\ \mu>\frac{1}{2}$.

But this deviation profit is only attainable if p^{AR} induces an indifferent consumer $\tilde{x} \in \left(\frac{1}{2}, 1\right)$. This is the case if and only if:

$$\begin{cases} p^{AR} > p^{B+} + \frac{\alpha}{2} - 1 \\ p^{AR} < p^{B+} - \frac{\alpha}{2} + \alpha\mu \end{cases} \Leftrightarrow \begin{cases} \alpha < \frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)} \\ \alpha < \frac{2+\mu}{3\mu(1-\mu)}. \end{cases}$$

The second condition is not restrictive. We can exclude, therefore, the existence of an equilibrium when $\alpha < \frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}$.

When, $\alpha > \max\left\{\frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}, \frac{2}{3\mu}\right\}$, the most profitable deviation is to choose the price, p^{A1}, that induces $\tilde{x} = 1$. From (3.4), $p^{A1} = p^{B+} + \frac{\alpha}{2} - 1 = \frac{2-4\mu}{3\mu}$. The corresponding profit is $\pi^{A1} = \frac{2-4\mu}{3\mu}$.

A necessary condition for profit maximization when $\alpha > \max\left\{\frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}, \frac{2}{3\mu}\right\}$ is:

$$\begin{aligned} \frac{[2(1+\mu)-3\alpha\mu]^2}{36\mu(1-\alpha\mu)} \geq \frac{2-4\mu}{3\mu} &\Leftrightarrow [2(1+\mu)-3\alpha\mu]^2 \geq 24(1-2\mu)(1-\alpha\mu) \\ &\Leftrightarrow 9\mu^2\alpha^2 + (12\mu - 60\mu^2)\alpha + 4\mu^2 + 56\mu - 20 \geq 0. \end{aligned}$$

Combining this condition with Assumption 3.1, we obtain:

$$\alpha \in \left(\max\left\{\frac{(2-\mu)(4\mu-1)}{3\mu(1-\mu)}, \frac{2}{3\mu}\right\}, \frac{10\mu - 2\sqrt{6}(1-2\mu) - 2}{3\mu}\right],$$

which is empty. □

Lemma 8.

If $\mu < \frac{1}{2}$, there exists no equilibrium with $\tilde{x} = \frac{1}{2}$.

Proof. The strategy of the proof is to show that $\tilde{x} = \frac{1}{2}$ is not compatible with profit-maximization by platform A.

From (3.3), an equilibrium with $\tilde{x} = \frac{1}{2}$ would have to be such that:

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{2}\left[1 + \alpha(D^A - D^B) - (p^A - p^B)\right] \\
\Leftrightarrow\ \alpha\left(\frac{\mu}{2} - \frac{1-\mu}{2}\right) &- p^A + p^B = 0 \\
\Leftrightarrow\ p^B &= p^A + \alpha\left(\frac{1-2\mu}{2}\right).
\end{aligned}$$

From (3.4), if $\tilde{x} \geq \frac{1}{2}$, the profit of platform A is given by:

$$\pi^A\big|_{\tilde{x}\geq\frac{1}{2}} = 2p^A\frac{2(1-\mu)(p^B - p^A) + 2\mu - \alpha(1-\mu)}{4\left[1-\alpha(1-\mu)\right]},$$

therefore, the derivative with respect to p^A is:

$$\left.\frac{\partial\pi^A}{\partial p^A}\right|_{\tilde{x}\geq\frac{1}{2}} = \frac{4(1-\mu)(p^B - p^A) + 4\mu - 2\alpha(1-\mu) - 4p^A(1-\mu)}{4\left[1-\alpha(1-\mu)\right]}.$$

For the profit-maximizing price, p^A, to imply $\tilde{x} = \frac{1}{2}$, it is necessary that the derivative above is non-negative, i.e., that:

$$\begin{aligned}
4(1-\mu)(p^B - 2p^A) + 4\mu - 2\alpha(1-\mu) &\geq 0 \\
\Leftrightarrow\ 4(1-\mu)\left[\alpha\left(\frac{1-2\mu}{2}\right) - p^A\right] + 4\mu - 2\alpha(1-\mu) &\geq 0 \\
\Leftrightarrow\ -4\alpha\mu(1-\mu) - 4(1-\mu)p^A + 4\mu &\geq 0 \\
\Leftrightarrow\ p^A \leq \frac{\mu}{1-\mu}\left[1-\alpha(1-\mu)\right]&.
\end{aligned} \tag{3.17}$$

From (3.4), if $\tilde{x} \leq \frac{1}{2}$, the profit of platform A is given by:

$$\pi^A\big|_{\tilde{x}\leq\frac{1}{2}} = 2p^A\frac{2\mu(p^B - p^A) + \mu(2-\alpha)}{4\left(1-\alpha\mu\right)},$$

therefore, the derivative with respect to p^A is:

$$\left.\frac{\partial\pi^A}{\partial p^A}\right|_{\tilde{x}\leq\frac{1}{2}} = \frac{4\mu(p^B - p^A) + 2\mu\left(2-\alpha\right) - 4\mu p^A}{4\left(1-\alpha\mu\right)}.$$

This derivative must be non-positive for the profit-maximizing price, p^A, to imply $\tilde{x} = \frac{1}{2}$. This occurs if and only if:

$$
\begin{aligned}
4(p^B - 2p^A) + 2\,(2-\alpha) &\leq 0 \\
\Leftrightarrow\; 4\left[\alpha\left(\frac{1-2\mu}{2}\right) - p^A\right] + 4 - 2\alpha &\leq 0 \\
\Leftrightarrow\; 2\alpha\,(1-2\mu) - 4p^A + 4 - 2\alpha &\leq 0 \\
\Leftrightarrow\; p^A \geq 1 - \alpha\mu. &
\end{aligned} \tag{3.18}
$$

But notice that (3.18) is incompatible with (3.17), because:

$$
\begin{aligned}
1 - \alpha\mu &\leq \frac{\mu}{1-\mu}\left[1 - \alpha(1-\mu)\right] \\
&\Leftrightarrow\; 1 - \alpha\mu \leq \frac{\mu}{1-\mu} - \alpha\mu \\
&\Leftrightarrow\; 1 - \mu \leq \mu,
\end{aligned}
$$

which is only true for $\mu = \frac{1}{2}$. □

Proof of Proposition 3

From Lemmas 7 and 8, we know that an interior equilibrium must be such that $\tilde{x} \in \left(\frac{1}{2}, 1\right)$.

(i) First-order conditions for profit-maximization

Considering the demand functions restricted to $\tilde{x} \in \left[\frac{1}{2}, 1\right]$, which are given by (3.7), the first-order conditions for profit maximization are:

$$
\begin{cases}
\frac{\partial \pi^A}{\partial p^A} = 0 \\
\frac{\partial \pi^B}{\partial p^B} = 0.
\end{cases}
$$

Solving this system of equations, we obtain the equilibrium prices from Proposition **3**.

(ii) Local second-order conditions for profit-maximization

To check that the equilibrium prices satisfy the local second-order condition, calculate:

$$
|H_1| = \frac{\partial^2 \pi^A}{\partial {p_1^A}^2} = \frac{-1+\mu}{1-\alpha^2(1-\mu)^2}.
$$

For the local second-order condition to be satisfied, it is necessary that:

$$|H_1| < 0 \;\Leftrightarrow\; \alpha < \frac{1}{1-\mu},$$

which is Assumption 3.1.

Computing the second-order cross partial derivatives we obtain:

$$\frac{\partial^2 \pi^A}{\partial p_1^A \partial p_2^A} = \frac{\partial^2 \pi^A}{\partial p_2^A \partial p_1^A} = \frac{-\alpha(1-\mu)^2}{1-\alpha^2(1-\mu)^2}.$$

Given symmetry:

$$\frac{\partial^2 \pi^A}{\partial {p_1^A}^2} = \frac{\partial^2 \pi^A}{\partial {p_1^A}^2}.$$

Therefore:

$$|H_2| = \left(\frac{\partial^2 \pi^A}{\partial {p_1^A}^2}\right)^2 - \left(\frac{\partial^2 \pi^A}{\partial p_1^A \partial p_2^A}\right)^2 = \frac{(1-\mu)^2 - \alpha^2(1-\mu)^4}{\left[1-\alpha^2(1-\mu)^2\right]^2}.$$

The local second-order condition is satisfied if:

$$|H_2| > 0 \;\Leftrightarrow\; 1-\alpha^2(1-\mu)^2 > 0 \;\Leftrightarrow\; \alpha < \frac{1}{1-\mu},$$

which is Assumption 3.1.

(iii) Interiority of the indifferent consumer

The indifferent consumer is located at:

$$\tilde{x}^* = \frac{1}{2} - \frac{\alpha(1-2\mu)}{12\left[1-\alpha(1-\mu)\right]} + \frac{1-2\mu}{6(1-\mu)}.$$

We must have $\tilde{x}^* \geq \frac{1}{2}$ (otherwise, this would not be an equilibrium):

$$\tilde{x}^* \geq \frac{1}{2} \;\Leftrightarrow\; \frac{1}{1-\mu} \geq \frac{\alpha}{2\left[1-\alpha(1-\mu)\right]} \;\Leftrightarrow\; \alpha \leq \frac{2}{3(1-\mu)}. \tag{3.19}$$

We also check that $\tilde{x}^* \leq 1$ (otherwise platform A would benefit from slightly increasing its price):

$$\tilde{x}^* \leq 1 \;\Leftrightarrow\; \frac{-2+\mu}{6(1-\mu)} \leq \frac{\alpha(1-2\mu)}{12\left[1-\alpha(1-\mu)\right]},$$

which is obviously true.

(iv) Global profit-maximization

Since the profit function of platform A is not globally quasi-concave, we must check that (given p^{B*}), the local maximizer of its profit function, p^{A*}, is a global maximum. For this to be true, it is necessary that:

$$\pi^A\left(p^{A*}, p^{B*}\right) \geq \pi^A\left(p^A, p^{B*}\right),\ \forall p^A. \tag{3.20}$$

Denote by p^{AL} the local maximum of $\pi^A\left(\cdot, p^{B*}\right)$ in the branch that leads to $\tilde{x} \in \left[0, \frac{1}{2}\right]$. From (3.4), in this branch, the profit function is given by:

$$\pi^A\left(p^A, p^{B*}\right) = 2p^A\left[\tfrac{\mu}{2(1-\alpha\mu)}\left(p^{B*} - p^A + 1 - \tfrac{\alpha}{2}\right)\right]. \tag{3.21}$$

The first-order condition for profit maximization in this branch yields:

$$\begin{aligned} & \tfrac{\partial \pi^A\left(p^A, p^{B*}\right)}{\partial p^A} = 0 \\ \Leftrightarrow\ & \tfrac{\mu}{2(1-\alpha\mu)}\left(p^{B*} - p^{AL} + 1 - \tfrac{\alpha}{2}\right) = p^{AL}\tfrac{\mu}{2(1-\alpha\mu)} \\ \Leftrightarrow\ & 2p^{AL} = p^{B*} + 1 - \tfrac{\alpha}{2}. \end{aligned}$$

From the equilibrium price p^{B*} of Proposition **3**, we obtain:

$$p^{AL} = \tfrac{5-4\mu}{6(1-\mu)} - \tfrac{\alpha}{2}.$$

This price is in the domain that leads to $\tilde{x} \in \left(0, \frac{1}{2}\right)$ if and only if:

$$\begin{cases} p^{AL} < p^{B*} - \frac{\alpha}{2} + 1 \\ p^{AL} > p^{B*} - \frac{\alpha}{2} + \alpha\mu \end{cases} \Leftrightarrow\ \alpha < \frac{5-4\mu}{3(1-\mu)},$$

which holds under condition (3.19).

Substituting p^{AL} in expression (3.21), we obtain:

$$\pi^A\left(p^{AL}, p^{B*}\right) = \tfrac{\mu}{(1-\alpha\mu)}\left[\tfrac{5-4\mu}{6(1-\mu)} - \tfrac{\alpha}{2}\right]^2. \tag{3.22}$$

Comparing this deviation profit with the equilibrium profit from Proposition 3, we conclude that the equilibrium condition (3.20) holds if and only if:

$$\frac{\left[2(1+\mu) - 3\alpha(1-\mu)\right]^2}{36(1-\mu)\left[1-\alpha(1-\mu)\right]} \geq \frac{\mu}{(1-\alpha\mu)}\left[\frac{5-4\mu}{6(1-\mu)} - \frac{\alpha}{2}\right]^2.$$

This condition is equivalent to non-negativity of the following polynomial:

$$9(1-\mu)^2\alpha^2 - 3(1-\mu)(4-\mu)\alpha + 4 - 5\mu \geq 0.$$

The roots of the polynomial are:

$$\alpha_c = \frac{4-\mu}{6\,(1-\mu)} \pm \frac{\sqrt{12\mu+\mu^2}}{6\,(1-\mu)}.$$

The equilibrium condition holds for α lower than the inferior root and for α greater than the superior root. However, while any α lower than the inferior root satisfies condition (3.19), any α greater than the superior root violates condition (3.19).

We conclude, therefore, that (3.20) is satisfied if and only if:

$$\alpha \leq \frac{4-\mu}{6\,(1-\mu)} - \frac{\sqrt{12\mu+\mu^2}}{6\,(1-\mu)}.$$

□

Proof of Proposition 5

An equilibrium with tipping in favor of platform B requires that $p^A = 0$ and $\tilde{x} = 0$. From the expression for the indifferent consumer (3.3), we obtain:

$$0 = 1 - \frac{\alpha}{2} + p^B \;\Leftrightarrow\; p^B = \frac{\alpha}{2} - 1.$$

Analogously, tipping in favor of platform A requires: $p^B = 0$ and $\tilde{x} = 1$. This implies, from (3.3):

$$2 = 1 + \frac{\alpha}{2} - p^A \;\Leftrightarrow\; p^A = \frac{\alpha}{2} - 1.$$

If $\alpha \leq 2$, prices are null, which implies that a small unilateral increase is profitable. □

3.6.3 Demand uniqueness at the interior equilibrium

Here, we investigate the existence of multiple possible demands for given prices, i.e., the existence of multiple equilibria in game in which consumers, for given prices, choose which platform to join.

Platform A can capture all demand if and only if it can attract the consumer that is located at $x = 1$:

$$u^A(1) \geq u^B(1) \;\Leftrightarrow\; V + \frac{\alpha}{2} - p^A - 1 \geq V + 0 - p^B - 0 \;\Leftrightarrow\; p^B - p^A \geq 1 - \frac{\alpha}{2}.$$

Similarly, platform B can capture all demand if and only if it can attract the consumer that is located at $x = 0$:

$$u^B(0) \geq u^A(0) \;\Leftrightarrow\; p^A - p^B \geq 1 - \frac{\alpha}{2}.$$

For the interior equilibrium prices, we have:

$$p^{B*} - p^{A*} = \frac{1 - 2\mu}{3(1 - \mu)}.$$

The interior equilibrium demand is unique if and only if:

$$\frac{1 - 2\mu}{3(1 - \mu)} < 1 - \frac{\alpha}{2} \;\Leftrightarrow\; \alpha < \frac{2(2 - \mu)}{3(1 - \mu)},$$

which is always satisfied. □

3.6.4 Characterization of the interior equilibrium

3.6.4.1 Impact of α on market shares ($\frac{\partial D^{A*}}{\partial \alpha} < 0$). To verify that $\frac{\partial D^{A*}}{\partial \alpha} < 0$, calculate:

$$\frac{\partial D^{A*}}{\partial \alpha} = \frac{-3(1 - \mu)12\left[1 - \alpha(1 - \mu)\right] + 12(1 - \mu)\left[2(1 + \mu) - 3\alpha(1 - \mu)\right]}{144\left[1 - \alpha(1 - \mu)\right]^2},$$

and notice that $\frac{\partial D^{A*}}{\partial \alpha} < 0$ if and only if:

$$36\left[1 - \alpha(1 - \mu)\right] > 12\left[2(1 + \mu) - 3\alpha(1 - \mu)\right] \;\Leftrightarrow\; \mu < \frac{1}{2}.$$

3.6.4.2 Impact of μ on profits ($\frac{\partial \pi^{B*}}{\partial \mu}$).

From π^{B*} of Proposition (3), we can calculate:

$$\frac{\partial \pi^{B*}}{\partial \mu} = \frac{[4-2\mu-3\alpha(1-\mu)]\,[2\mu-\alpha(1-\mu)]}{36\,[1-\alpha(1-\mu)]^2\,(1-\mu)^2}.$$

Therefore:

$$\begin{aligned}\frac{\partial \pi^{B*}}{\partial \mu} > 0 \;&\Leftrightarrow\; [4-2\mu-3\alpha(1-\mu)]\,[2\mu-\alpha(1-\mu)] > 0\\ &\Leftrightarrow\; 3(1-\mu)^2\alpha^2 - 4(1-\mu^2)\alpha + 8\mu - 4\mu^2 > 0\end{aligned}$$

The roots of this second-degree polynomial in α are:

$$r_1 = \frac{2\mu}{1-\mu} \quad \text{and} \quad r_2 = \frac{4-2\mu}{3(1-\mu)}.$$

Since the value of the second root is outside the domain of existence of equilibrium, we conclude that:

$$\begin{cases} \frac{\partial \pi^{B*}}{\partial \mu} > 0, & \text{if } \alpha \in \left(0, \frac{2\mu}{1-\mu}\right) \\ \frac{\partial \pi^{B*}}{\partial \mu} < 0, & \text{if } \alpha \in \left(\frac{2\mu}{1-\mu}, \frac{4-\mu}{6(1-\mu)} - \frac{\sqrt{12\mu+\mu^2}}{6(1-\mu)}.\right). \end{cases}$$

3.6.4.3 Impact of α on profits ($\frac{\partial \pi^{B*}}{\partial \alpha} < 0$).

From π^{B*} of Proposition (3), we can calculate:

$$\frac{\partial \pi^{B*}}{\partial \alpha} = \frac{-\,[4-2\mu-3\alpha(1-\mu)]\,[2(1+\mu)-3\alpha(1-\mu)]}{36\,[1-\alpha(1-\mu)]^2}.$$

Therefore:

$$\frac{\partial \pi^{B*}}{\partial \alpha} > 0 \;\Leftrightarrow\; [4-2\mu-3\alpha(1-\mu)]\,[2(1+\mu)-3\alpha(1-\mu)] < 0.$$

Since $\alpha \leq \frac{2}{3(1-\mu)}$, the second parcel is surely positive. Thus:

$$\frac{\partial \pi^{B*}}{\partial \alpha} > 0 \;\Leftrightarrow\; 4-2\mu-3\alpha(1-\mu) < 0 \;\Leftrightarrow\; \alpha > \frac{4-2\mu}{3(1-\mu)},$$

which is always false.

3.6.5 Social welfare maximization

If $\tilde{x} \in \left[0, \frac{1}{2}\right]$, social welfare is given by:

$$\begin{aligned}W(\tilde{x}) &= 2\left\{\int_0^{\tilde{x}} \left(V + \alpha D^A - x\right)\mu dx + \int_{\tilde{x}}^{\frac{1}{2}} \left[V + \alpha D^B - (1-x)\right]\mu dx\right.\\ &\left.\quad + \int_{\frac{1}{2}}^{1} \left[V + \alpha D^B - (1-x)\right](1-\mu)dx\right\}\\ &= V - \frac{1}{4} - \frac{\mu}{2} + 2\mu\tilde{x}(1-\tilde{x}) + 2\alpha\mu\tilde{x}(D^A - D^B) + \alpha D^B.\end{aligned}$$

Since $D^A = \mu\tilde{x}$ and $D^B = \frac{1}{2} - \mu\tilde{x}$, we obtain:

$$W(\tilde{x}) = V - \frac{1}{4} + \frac{\alpha}{2} - \frac{\mu}{2} + \tilde{x}^2\left[-2\mu(1-2\alpha\mu)\right] + \tilde{x}\left[2\mu(1-\alpha)\right].$$

If $\alpha < 1$, this is a concave function, with maximum at:

$$\tilde{x}^{FB} = \frac{1-\alpha}{2\left(1-2\alpha\mu\right)}.$$

With $\tilde{x} = \tilde{x}^{FB}$, welfare is given by:

$$W^{FB} = V - \frac{1}{4} + \frac{\alpha}{2} - \frac{\mu}{2} + \frac{\mu(1-\alpha)^2}{2(1-2\alpha\mu)}.$$

Now let us check that welfare cannot be greater for some $\tilde{x} \in \left(\frac{1}{2}, 1\right]$. At $\tilde{x} = 1$, welfare is lower than at $\tilde{x} = 0$, because total transportation costs are higher. Therefore, we only need to check the possible interior maximum.

With $\tilde{x} \in \left(\frac{1}{2}, 1\right]$, welfare is given by:

$$\begin{aligned}W(\tilde{x}) &= 2\left\{\int_0^{\frac{1}{2}} (V + \alpha D^A - x)\mu dx + \int_{\frac{1}{2}}^{\tilde{x}} (V + \alpha D^A - x)(1-\mu)dx\right.\\ &\left.\quad + \int_{\tilde{x}}^{1} (V + \alpha D^B - (1-x))(1-\mu)dx\right\}.\end{aligned}$$

Replacing $D^A = \frac{\mu}{2} + (\tilde{x} - \frac{1}{2})(1-\mu)$ and $D^B = (1-\tilde{x})(1-\mu)$, and simplyfying, we obtain:

$$\begin{aligned}W(\tilde{x}) &= V - \frac{3}{4} + \frac{\mu}{2} + \frac{5\alpha}{2} - 6\alpha\mu + 4\alpha\mu^2 - 2\tilde{x}^2[1 - 2\alpha(1-\mu)](1-\mu)\\ &\quad + 2\tilde{x}(1-\mu)[1 - \alpha(3-4\mu)].\end{aligned}$$

For $\alpha < \frac{1}{2(1-\mu)}$, this is a concave function, with maximum at:

$$\tilde{x}^{+} = \frac{1-\alpha(3-4\mu)}{2-4\alpha(1-\mu)}.$$

This potential maximizer is inside the domain if and only if:

$$\alpha(3-4\mu) \leq 2\alpha(1-\mu) \;\Leftrightarrow\; 3-4\mu \leq 2-2\mu \;\Leftrightarrow\; 1 \leq 2\mu,$$

which is false. The constrained maximum is not interior, therefore, it is lower than the constrained maximum at $\tilde{x} \in \left[0, \frac{1}{2}\right]$.

4.0 ENDOGENOUS QUALITY AND GROUP DISCRIMINATION IN TWO-SIDED MARKETS

Abstract. We consider a model of vertical and horizontal differentiation in two-sided markets. We find that, as the degree of vertical differentiation increases, the equilibrium prices, market shares and profits of the low-quality (high-quality) platform decrease (increase). We introduce a two-stage game where platforms first choose their quality levels and then compete in prices. We fully characterize the asymmetric equilibrium where the high-quality platform charges a higher price and conquers the majority of the market. We also propose an extension where only the high-quality platform chooses its investment level on quality and the quality of the high-quality platform is different between the two sides of the market. We find that divide and conquer strategies arise in equilibrium because the low-quality platform charges a price below its marginal cost in the side where the high-quality platform has a lower cost of quality provision.

Keywords: Two-sided Markets, Horizontal differentiation, Vertical differentiation, Tipping, Group Discrimination.

JEL Classification Numbers: D42, D43, L10.

4.1 INTRODUCTION

Usually, researchers use the Hotelling line to perform a duopoly price competition analysis in two-sided markets, in which under pure horizontal differentiation, at the same price level, a fraction of consumers of a side j travels to a platform located at one extreme of the segment and the remaining fraction of consumers travels to the opposite platform. However, in the case of pure vertical differentiation, the platforms have different qualities. Thus, at the same price level, all side j consumers prefer one platform relatively to the other.

Vertical differentiation is discussed since the canonical contribution of Mussa and Rosen (1978) [85], among others. Economides (1989) [39] allows for quality variations in a duopoly of locationally differentiated products à la Hotelling (1929) [68] and shows that in a sequential game of variety choice and subsequent quality and price choice there exists only maximal variety differentiation equilibria in pure strategies. In addition, he also finds that maximal variety differentiation is the perfect pure strategies equilibrium of a sequential game of variety choice followed by quality choice and later by price choice. However, in both games there is minimal quality differentiation at equilibrium. Thus, the author emphasizes the so called "Min-Max" differentiation principle.

The integration of simultaneous horizontal and vertical differentiation in two-sided markets is a recent matter in two-sided markets.

Viecens (2006) [111] constructs a model of platform price competition where, first platforms endogenously decide the quality of their 'access service' and second, each group exhibits preferences not only about the number of agents in the opposite group, but also about their type or quality. She examines the set of conditions under which, in spite of the inter-group externalities, more than one platform survives in equilibrium since it shows that when quality is endogenously determined by the choices of agents these platforms could be asymmetric[1].

Lin, Li and Whinston (2011) [70] examine a platform's optimal pricing strategy while considering seller-side innovation decisions and price competition. They find that the plat-

[1] We intend to perform a similar analysis as an extension of our benchmark model but in a setting à la Armstrong (2006) [9], where the platforms are horizontally placed at the end extremes of the unit line, for exogenous reasons and then, platforms decisions are simultaneously made: firstly, (i) quality, and secondly (ii) prices, in line with Sanjo (2007) [100] and Sanjo (2009) [101].

form's optimal strategy may be to charge or subsidize buyers depending on the degree of variation in the buyers' willingness-to-pay for quality. They also find that when all sellers innovate, there exists a parameterization under which a higher seller-side access fee stimulates innovation.

Njoroge et al. (2009) [88] and Njoroge et al. (2010) [89], focusing on online markets, study duopoly competition between two interconnected Internet Service Providers (ISP) that compete in quality and prices for both Content Providers (CP) and consumers. In this work, platforms first choose quality levels from a bounded interval and in the subsequent stages compete in prices for both CP's and consumers. They show that a Sub-game Perfect Equilibrium for the whole game exists and characterize all the equilibrium choices of the quality game, in the sense that the equilibria involve either maximal differentiation or partial differentiation.

Regarding platforms' quality, Ponce (2012) [91] studies the consequences of the transition from an "investor-pays" model to an "issuer-pays" model on the quality standard of credit ratings chosen by the agency. He finds that such a transition is likely to generate a degradation of quality, which may fall below the socially efficient level. Gabszewicz and Wauthy (2014) [58] model platform competition in a market where products are also characterized by inter-group externalities. Consumers differ in their externalities valuation. With exogenous symmetry between both sides of the market, they show that platform competition induces a vertical differentiation structure that allows for the co-existence of asymmetric platforms in equilibrium.

Our manuscript explores the possibility of tipping in the presence of quality asymmetries. Gold (2010) [62] analyzes this issue but in a context of cost asymmetries and finds that the magnitude of the effects of inter-platform cost differences on equilibrium membership depends on how the asymmetric costs are borne: in regimes with high network effects, differences in the platforms' costs of serving customers have only limited impact, while the impact of cost differences for the market sides are less clear. Real world examples of tipping are plentiful[2]. With the influence of inter-group externalities, one member of a particular side announcing which platform will join could induce standardization in the market.

[2] See Gold (2010) [62] for examples where tipping occurs in markets with network effects.

We also analyze group discrimination in the following way: the two sides of the market differ on the perception of the degree of vertical differentiation between the platforms. To the best of our knowledge, no prior manuscript studies this issue. However, there exists real world evidence of this feature. Consider the console industry and the smartphone industry. One side of the market is composed by the end-users. The other side is composed by software developers. It is realist to say that the perceived quality gap between the active platforms on the market (for instance, Playstation and X-Box in the consoles industry and the I-Phone and Samsung Galaxy in the smartphone industry, respectively) is distinct between the two sides of the market. Software developers are essentially worried with the performance of the operating systems (IOS in the I-Phone and Android in the Samsung Galaxy). End-users may be worried with the same topic. However, end-users may value extra issues such as the screen size, the cell phone design, the camera resolution and so on, which are not generally on the scope of developers. By the same argument, developers prefer an easier technical language in order to be able to program more complex applications. Nonetheless, it might be the case that end-users dislike the final product of the platform.

This manuscript is relevant to the literature of two-sided markets. As a motivating example, consider the case of newspapers. The sides of the market are composed by readers and advertisers, respectively. Also assume that all the readers are "ad-lovers" such that the cross network effect is positive between the two sides of the market. There exist two types of intermediary newspapers: a low-quality newspaper whose content is free for readers (free newspapers) (and, thus, the access price charged by this low-quality newspaper to readers is below or equal to its marginal cost) and a high-quality newspaper whose content is not free for readers (non-free newspapers) (and, thus, the access price charged by the high-quality newspaper to readers is above or equal to its marginal cost). Both platforms charge positive access prices to advertisers. For simplicity, assume that only the high-quality newspaper invests in quality provision on advertisers (side 1) and on readers (side 2). Thus, for the low-quality newspaper it is too costly to provide quality on both sides of the market.

In a two-sided market, a divide and conquer strategy emerges when a certain side is subsidized and to the opposite side is charged a positive price (Caillaud and Jullien (2003) [21]). The reason exposed in the literature for such strategy to appear in equilibrium relies on the fact that the inter-group externality is stronger in one of the sides of the market. The side that is subsidized is the side with the stronger inter-group externality (Caillaud and Jullien (2003) [21]).

Our manuscript assumes the same level of the inter-group externality between both sides of the market. However, we obtain a divide and conquer strategy conducted only by the low-quality newspaper in equilibrium. Such strategy emerges because we consider that the sides of the market have a different perception of the quality provision of the high-quality newspaper instead of considering an asymmetric strength on the inter-group externality between the two sides of the market.

In our example, this implies that we consider that it is more costly to the high-quality platform to provide quality to advertisers (side 1) relatively to readers (side 2). Then, the low-quality newspaper reacts subsidizing the readers' side (side 2), the side where is less costly for the high-quality newspaper to provide quality and, by the network effect, this side will attract advertisers (the opposite side 1) to the low-quality newspaper. Our argument tries to explain how an intermediary counterbalances its lower quality, by adopting a divide and conquer strategy where the side that is subsidized is the one where it is less costly for the high-quality rival to provide quality.

The paper is organized as follows. Section 4.2 presents the model and the correspondent subsections fully characterize the interior equilibrium, tipping and welfare analysis. Section 4.3 determines the social optimum outcome. Section 4.4 provides the analysis with a pre-investment stage and section 4.5 generalizes the initial model allowing the stand-alone values to be different between the two sides of the market. Finally, section 4.4 concludes. All details on proofs and lemmas are relegated to Appendix.

4.2 THE MODEL

We consider a two-sided market with two platforms, A and B, which are horizontally and vertically differentiated. Both platforms are exogenously located at the extremes of a linear city: platform A is located at $x = 0$ while platform B is located at $x = 1$.

We normalize marginal costs to zero[3]. There is a continuum of singlehoming consumers on each of the two sides, 1 and 2, that inelastically demand one unit of the service provided by the platform. We follow Armstrong (2006) [9] in considering linear inter-group externalities and Serfes and Zacharias (2012) [104] in assuming symmetry between the two sides of the market.

Horizontal differentiation is captured by linear transportation costs (Hotelling (1929) [68]). For technical convenience, the transportation cost parameter is the same for both sides of the market, t (Serfes and Zacharias (2012) [104]).

Vertical differentiation is captured by the fact that consumers value platforms differently: platform B by the amount V^B and platform A by the amount V^A.[4] We assume that platform B is the high-quality platform: $V^B \geq V^A$. To simplify the notation assume $V^B - V^A \equiv q$.

The utility of an agent of side j, $j \in \{1,2\}$ that is located at $x_j \in [0,1]$ and chooses platform $i \in \{A, B\}$ is given by:

$$u_j^A(x_j) = V^A + \alpha D_k^A - p_j^A - tx_j; \tag{4.1}$$

$$u_j^B(x_j) = V^B + \alpha D_k^B - p_j^B - t(1 - x_j), \tag{4.2}$$

where α measures the strength of the inter-group externality, k is the other side ($k = 2$ if $j = 1$ and $k = 1$ if $j = 2$), D_k^i is the number of consumers of side k that join platform i and p_j^i is the access price charged by platform i to consumers of side j. This constitutes our benchmark case, in which there is no group discrimination (on the perceived quality of the platforms) between the two sides of the market.

The timing of the game is the following: in the first stage, platforms simultaneously set access prices for both sides. In the second stage, agents simultaneously choose which platform to join and their respective payoffs are determined. The game is solved by backward induction.

Throughout the manuscript, we assume that the degree of product differentiation is sufficiently high relatively to the intensity of the inter-group externality for the model to be well behaved.

[3]The results would be qualitatively the same if, instead, we assumed the same constant marginal costs on both platforms.

[4]Both V^A and V^B are sufficiently high so that the market is totally covered.

Assumption 4.1 (Weak inter-group externality) *The inter-group externality is relatively weak:* $\alpha < t$.

4.2.1 Demand and profits as a function of prices

We assume that the pure singlehoming agents have rational expectations relatively to the platform choices of the other agents. This implies that all consumers to the left of the indifferent consumer choose platform A and all consumers to the right choose platform B.

Thus, the demand for platform A from each consumer side as a function of the location of the indifferent consumer is given by:

$$D_j^A(\widetilde{x}_j) = \begin{cases} 1 & \widetilde{x}_j \geq 1; \\ \widetilde{x}_j & 0 \leq \widetilde{x}_j \leq 1; \\ 0 & \widetilde{x}_j \leq 0. \end{cases}$$

A consumer that is indifferent between platform A and B and correctly anticipates the behavior of the agents in the other side of the market must be located at:

$$\widetilde{x}_j = \frac{1}{2} + \frac{\alpha(p_k^B - p_k^A) + t(p_j^B - p_j^A)}{2(t-\alpha)(t+\alpha)} - \frac{q}{2(t-\alpha)}. \tag{4.3}$$

As a function of prices, the demand for platform A is given by:

$$D_j^A(p_j^A, p_j^B) = \begin{cases} 1, \quad tp_j^A + \alpha p_k^A \leq tp_j^B + \alpha p_k^B - (t+\alpha)(q+t-\alpha); \\ \\ \frac{1}{2} + \frac{\alpha(p_k^B - p_k^A) + t(p_j^B - p_j^A)}{2(t-\alpha)(t+\alpha)} - \frac{q}{2(t-\alpha)}, \quad \begin{array}{l} tp_j^B + \alpha p_k^B - (t+\alpha)(q+t-\alpha) \\ \leq tp_j^A + \alpha p_k^A \leq \\ tp_j^B + \alpha p_k^B - (t+\alpha)(q-t+\alpha); \end{array} \\ \\ 0, \quad tp_j^A + \alpha p_k^A \geq tp_j^B + \alpha p_k^B - (t+\alpha)(q-t+\alpha). \end{cases}$$

Since total demand is inelastic, the demand for platform B from each consumer side is $D_j^B(p_j^A, p_j^B) = 1 - D_j^A(p_j^A, p_j^B)$.

The profit of each platform $i \in \{A, B\}$, is given by $\pi^i(p_j^A, p_j^B) = \sum_{j=1}^{2} p_j^i D_j^i(p_j^A, p_j^B)$.

4.2.2 Interior equilibrium

Firstly, we ensure that both platforms charge positive prices in equilibrium and then we fully characterize the unique interior equilibrium.

Assumption 4.2 (Positive prices and market shares) *The inter-group externality is sufficiently weak:* $\alpha < t - \frac{q}{3}$.

Proposition 9. ***(Equilibrium without discrimination)***

For $\alpha \in \left[0, t - \frac{q}{3}\right)$, *the interior equilibrium prices are given by:*

$$\begin{cases} p_j^{A*} = t - \alpha - \frac{q}{3}; \\ p_j^{B*} = t - \alpha + \frac{q}{3}. \end{cases} \tag{4.4}$$

The interior equilibrium market shares are given by:

$$\begin{cases} D_j^{A*} = \frac{1}{2} - \frac{q}{6(t-\alpha)}; \\ D_j^{B*} = \frac{1}{2} + \frac{q}{6(t-\alpha)}. \end{cases} \tag{4.5}$$

The platforms' equilibrium profits are given by:

$$\begin{cases} \pi^{A*} = \frac{[3(t-\alpha)-q]^2}{9(t-\alpha)}; \\ \pi^{B*} = \frac{[3(t-\alpha)+q]^2}{9(t-\alpha)}. \end{cases} \tag{4.6}$$

Proof. See Appendix 4.8.1. □

From Proposition 9, we conclude that the high-quality platform charges the higher equilibrium price. It will always conquer the majority of the agents on the market since $D_j^{B*} \geq \frac{1}{2}$, $\forall\ q > 0$ and, thus, gets higher profits.

The equilibrium outcomes are similar to those in Griva and Vettas (2011) [63], although their model is focused on simple network effects. Thus, we find that a two-sided market model coincides with a simple network effects model when we consider the same inter-group externality and transportation cost on both sides of the market.

4.2.3 Comparative statics

Comparative statics provide the impacts of the degree of vertical differentiation and of the inter-group externality on each of the equilibrium outcomes. As the degree of vertical differentiation (q) increases, the equilibrium price of the high-quality quality platform increases while the equilibrium price of the low-quality platform decreases. Both prices vary in opposite ways but with the same magnitude:

$$\frac{\partial p^{A*}}{\partial q} < 0 < \frac{\partial p^{B*}}{\partial q} \text{ and } \frac{\partial p^{B*}}{\partial q} = \left| \frac{\partial p^{A*}}{\partial q} \right|.$$

We also remark that as the quality gap between platforms increases, the equilibrium market share of the high-quality platform increases while the equilibrium market share of the low-quality platform decreases. Thus, as the degree of vertical differentiation increases the low-quality platform loses market share ($\frac{\partial D^{A*}}{\partial q} < 0$), and, therefore, also loses profits ($\frac{\partial \pi^{A*}}{\partial q} < 0$). However, since the market share of the high-quality platform increases ($\frac{\partial D^{B*}}{\partial q} > 0$), the high-quality platform will get more profits ($\frac{\partial \pi^{B*}}{\partial q} > 0$).

Prices decrease as the inter-group externality becomes stronger ($\frac{\partial p^{A*}}{\partial \alpha} = \frac{\partial p^{B*}}{\partial \alpha} = -1$). Market shares diverge as the strength of the inter-group externality increases ($\frac{\partial D^{A*}}{\partial \alpha} < 0 < \frac{\partial D^{B*}}{\partial \alpha}$). Thus, as the strength of the inter-group externality increases, the profit of platform A decreases, which is in line with Armstrong (2006) [9]. In the case of the high-quality platform B, the negative price effect dominates the positive demand effect since $\frac{\partial \pi^{B*}}{\partial \alpha} = -1 + \frac{q^2}{9(t-\alpha)^2}$, which is strictly negative under Assumption 4.2.

Lemma 10. ***(Impact of vertical differentiation)***

As the degree of vertical differentiation increases:

(i) the equilibrium prices and market shares of the low-quality (high-quality) platform decrease (increase);

(ii) the equilibrium profit of the low-quality (high-quality) platform decreases (increases).

Proof. The proof follows directly from the explanation above. □

4.2.4 Welfare analysis

The consumer surplus of each side equals:

$$CS_j = \int_0^{\widetilde{x}_j} u_j^A dx + \int_{\widetilde{x}_j}^{1} u_j^B dx.$$

which is equivalent to:

$$V^B - \widetilde{x}q + \alpha D^B - \alpha\widetilde{x}(D^B - D^A) - p^B + \widetilde{x}(p^B - p^A) - \frac{t}{2} + t\widetilde{x}(1-\widetilde{x}).$$

In equilibrium, the consumer surplus on each side of the market is given by:

$$CS_j^* = \frac{q + 2V^A + 3\alpha}{2} + \frac{tq^2}{36(t-\alpha)^2} - \frac{5}{4}t. \tag{4.7}$$

Lemma 11. ***(Vertical differentiation and consumer surplus)***

As the degree of vertical differentiation decreases (increasing V^A while keeping V^B constant), the consumer surplus increases if:

$$q < \frac{9(t-\alpha)^2}{t}. \tag{4.8}$$

Proof. See Appendix 4.8.2. □

Total welfare is obtained in a similar way but without taking into consideration the access prices. Total welfare is given by:

$$W^* = q + 2V^A + \alpha - \frac{t}{2} + \frac{q^2}{18(t-\alpha)^2}\left[t + 4\left(t-\alpha\right)\right]. \tag{4.9}$$

Lemma 12. ***(Vertical differentiation and welfare)***

As the degree of vertical differentiation decreases (increasing V^A while keeping V^B con- stant), *the total welfare increases if:*

$$q < \frac{9\left(t-\alpha\right)^2}{5t-4\alpha}. \tag{4.10}$$

Proof. See Appendix 4.8.2. □

4.2.5 Tipping

This section investigates tipping. Tipping corresponds to a case where one of the platforms conquers the whole market.

Lemma 13. ***(Characterization of tipping)***

(i) Tipping in favor of the high-quality platform B may occur for $\alpha > t - q$. In this case, the high-quality platform charges an access price on each side of the market equal to $p^{B(T)*} = \alpha - (t - q)$, the market share on both sides of the market is $D_1^{B(T)*} = D_2^{B(T)*} = 1$ and earns a total profit $\pi^{B(T)*} = 2\alpha - 2(t - q)$.

(ii) Tipping in favor of the low-quality platform A cannot occur for $\alpha < t$.

Proof. See Appendix 4.8.3. □

Lemma 13 shows that the low-quality platform A only captures the whole market for a sufficiently higher level of the inter-group externality relatively to the high-quality platform B.

However, is worth noting that even for $\alpha > t - q$, the high-quality platform not always monopolizes the market. Indeed, the high-quality platform has incentive to monopolize if and only if

$$\pi^{B(T)*} \geq \pi^{B(Interior)*}.$$

Proposition 14. ***(Tipping equilibrium)***

Tipping in favor of the high-quality platform is a market equilibrium for $\alpha > t - \frac{q}{3}$.

Proof. see Appendix 4.8.3. □

At the lower bound where the high-quality platform either tips the market or both platforms stay active in the market ($\alpha = t - \frac{q}{3}$ or, alternatively, $q = 3(t - \alpha)$) the price charged by the high-quality platform is $p^{B(T)*} = 2(t - \alpha)$ and the profit is equal to $\pi^{B(T)*} = 4(t - \alpha)$.

Notice that the argument exposed for the high-quality platform does not hold for the low-quality platform because the interior equilibrium region $\left(\alpha \leq t - \frac{q}{3}\right)$ and the tipping region that favors the low-quality platform $(\alpha > t + q)$ are mutually exclusive.

4.3 SOCIAL OPTIMUM

In this section, we consider a benevolent planner that is assumed to be able to choose the position of the indifferent consumer in order to maximize social welfare. Conditionally on $\tilde{x} \in [0, 1]$, social welfare as a function of $\tilde{x}$ is given by:

$$W(\tilde{x}) = V^B - \widetilde{x}q + \alpha D^B - \alpha\widetilde{x}\left(D^B - D^A\right) - \tfrac{t}{2} + t\widetilde{x}(1 - \widetilde{x}). \tag{4.11}$$

The maximum in $[0, 1]$ is attained at:

$$\tilde{x}^{FB} = \frac{1}{2} - \frac{q}{2t - 4\alpha}, \tag{4.12}$$

conditionally on $\alpha < \frac{t-q}{2}$.

Proposition 15. ***(Social optimum versus market equilibrium)***

(i) Let $\alpha < \frac{t-q}{2}$. In the socially optimal outcome, both platforms operate in the market, but with the high-quality platform capturing a higher market share than in the market equilibrium.

(ii) Let $\frac{t-q}{2} \leq \alpha < t$. In the socially optimal outcome, the high-quality platform captures the whole market.

Proof. *See Appendix 4.8.4.* □

Comparing the socially optimal outcome with the market equilibrium outcome, we conclude that the regulator aims to concentrate more consumers on the high-quality platform in comparison with the market equilibrium.

This is due to a bad internalization of the network effects by the intermediaries and also because the strategic competition in prices leads the platform with the highest expected quality to be the most expensive, which confirms Argenziano (2008) [7].

4.4 ENDOGENOUS QUALITY

To develop our analysis, we now introduce an investment stage, where the platforms have the possibility of endogenously determining their qualities. Each platform i has to invest $I(V^i)$ to obtain a quality level V^i. We assume that $I(V^i) = \frac{1}{2}\beta_i {V^i}^2$, following Economides (1989) [39], among others.

The parameter β_i measures the cost of quality provision of platform i. To keep platform B as the high-quality platform consider $\beta_A > \beta_B > 0$. Since β_i is a measurement of the cost of quality provision, the low-quality platform must put an extra-effort to produce the same quality level relatively to the high-quality platform.

Thus, *caeteris paribus*, in the first stage platforms choose their quality level, $V^i \geq 0$. The second stage remains unchanged. In the simultaneous non-cooperative stage of quality investment, we reach the following result.

Proposition 16. ***(Equilibrium with no discrimination and endogenous quality)***

If $\beta_B > \frac{4}{9(t-\alpha)}$, the unique subgame perfect pure-strategy equilibrium is such that the qualities, the prices, the market shares and the platforms' profits are given, respectively, by:

$$V^{B*} - V^{A*} = \frac{6(t-\alpha)\left(\beta_A - \beta_B\right)}{\beta_B\left[9(t-\alpha)\beta_A - 2\,\right] - 2\beta_A}; \tag{4.13}$$

$$\begin{cases} p^{A*} = \frac{\beta_A(t-\alpha)[9(t-\alpha)\beta_B - 4\,]}{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A}; \\ p^{B*} = \frac{\beta_B(t-\alpha)[9(t-\alpha)\beta_A - 4\,]}{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A}; \end{cases} \tag{4.14}$$

$$\begin{cases} D_j^{A*} = \frac{1}{2} - \frac{\beta_A - \beta_B}{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A}; \\ D_j^{B*} = \frac{1}{2} + \frac{\beta_A - \beta_B}{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A}; \end{cases} \tag{4.15}$$

$$\begin{cases} \pi^{A*} = \frac{\beta_A[9(t-\alpha)\beta_B - 4\,]^2[9(t-\alpha)\beta_A - 2\,]}{9\{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A\,\}^2}; \\ \pi^{B*} = \frac{\beta_B[9(t-\alpha)\beta_A - 4\,]^2[9(t-\alpha)\beta_B - 2\,]}{9\{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A\,\}^2}. \end{cases} \tag{4.16}$$

Proof. See Appendix 4.8.5. □

Since the low-quality platform spends an extra cost relatively to the high-quality platform to achieve the same quality level $(\beta_A > \beta_B)$, in equilibrium, quality, prices, market shares and profits are higher for the high-quality platform.

As the cost of quality provision of the high-quality platform increases, the degree of vertical differentiation between platforms is reduced. Then, the low-quality platform has incentives to increase its price and the high-quality platform to reduce it. The market shares tend to diverge with the low-quality platform gaining extra-consumers, since the quality gap is smaller. As a result, as β_B increases, we get a higher profit for the low-quality platform and a lower profit for the high-quality platform.

On the other hand, when the quality effort of the low-quality platform increases, the degree of vertical differentiation between platforms increases. Then, the high-quality platform has incentives to increase its price and the low-quality platform to reduce its price. The market shares tend to diverge with the high-quality platform now gaining extra-consumers since the quality gap is larger, from which follows a higher profit for the high-quality platform and a lower profit for the low-quality platform.

Tipping in favor of the high-quality platform occurs for $\beta_B < \frac{4}{9(t-\alpha)}$. Thus, when the cost of quality provision of the high-quality platform is sufficiently low, competition vanishes.

Lemma 17. ***(Impacts of the cost of quality production)***

The equilibrium prices, equilibrium market shares and equilibrium profit of platform $i \in \{A, B\}$ increase (decrease) as:

(i) β_i decreases (increases);

(ii) β_l increases (decreases),

with ($l = B$ if $i = A$ and $l = A$ if $i = B$).

Proof. See Appendix 4.8.5. ☐

For simplicity, let $V_1^B - V_1^A \equiv q_1$ and $V_2^B - V_2^A \equiv q_2$. Suppose w.l.o.g. that $q_1 > q_2 > 0$. The platforms' quality can, now, be different between the two sides of the market [5].

In this case, a consumer that is indifferent between platform A and B must be located at:

$$\widetilde{x}_j = \frac{1}{2} + \frac{\alpha(p_k^B - p_k^A) + t(p_j^B - p_j^A)}{2(t-\alpha)(t+\alpha)} - \frac{\alpha q_k + t q_j}{2(t-\alpha)(t+\alpha)}. \tag{4.17}$$

As a function of prices, the demand for platform A is given by:

$$D_j^A(p_j^A, p_j^B) = \begin{cases} 1, \quad tp_j^A + \alpha p_k^A \leq -(t+\alpha)(t-\alpha) + tp_j^B + \alpha p_k^B - \alpha q_k - tq_j; \\ \\ \frac{1}{2} + \frac{\alpha(p_k^B - p_k^A) + t(p_j^B - p_j^A)}{2(t-\alpha)(t+\alpha)} - \frac{\alpha q_k + t q_j}{2(t-\alpha)(t+\alpha)}, \quad -(t+\alpha)(t-\alpha) + tp_j^B + \alpha p_k^B - \alpha q_k - tq_j \\ \qquad \leq tp_j^A + \alpha p_k^A \leq \\ \qquad (t+\alpha)(t-\alpha) + tp_j^B + \alpha p_k^B - \alpha q_k - tq_j; \\ \\ 0, \quad tp_j^A + \alpha p_k^A \geq (t+\alpha)(t-\alpha) + tp_j^B + \alpha p_k^B - \alpha q_k - tq_j. \end{cases}$$

4.5.1 Interior equilibrium

Focusing on the interior equilibrium, we ensure that both platforms are active on the market.

Assumption 4.3 (Weak network effects with discrimination) *The inter-group externality is relatively weak:* $\alpha q_2 + t q_1 < 3(t-\alpha)(t+\alpha)$.

Below we fully characterize the unique interior equilibrium.

Proposition 18. ***(Equilibrium with discrimination)***

The interior equilibrium prices are given by:

$$\begin{cases} p_1^{A*} = t - \alpha - \frac{q_1}{3}; \quad p_2^{A*} = t - \alpha - \frac{q_2}{3}; \\ p_1^{B*} = t - \alpha + \frac{q_1}{3}; \quad p_2^{B*} = t - \alpha + \frac{q_2}{3}. \end{cases} \tag{4.18}$$

[5] All calculations are presented in Appendix 4.8.6.

The equilibrium market shares are given by:

$$\begin{cases} D_1^{A*} = \frac{1}{2} - \frac{\alpha q_2 + t q_1}{6(t-\alpha)(t+\alpha)}; \quad D_2^{A*} = \frac{1}{2} - \frac{\alpha q_1 + t q_2}{6(t-\alpha)(t+\alpha)}; \\ D_1^{B*} = \frac{1}{2} + \frac{\alpha q_2 + t q_1}{6(t-\alpha)(t+\alpha)}; \quad D_2^{B*} = \frac{1}{2} + \frac{\alpha q_1 + t q_2}{6(t-\alpha)(t+\alpha)}. \end{cases} \tag{4.19}$$

and the platforms' equilibrium profit are given by:

$$\begin{cases} \pi^{A*} = \frac{18t^3 - 18t^2\alpha - 6t^2 q_1 - 6t^2 q_2 - 18t\alpha^2 + t q_1^2 + t q_2^2 + 18\alpha^3 + 6\alpha^2 q_1 + 6\alpha^2 q_2 + 2\alpha q_1 q_2}{18(t^2-\alpha^2)} \\ \pi^{B*} = \frac{18t^3 - 18t^2\alpha + 6t^2 q_1 + 6t^2 q_2 - 18t\alpha^2 + t q_1^2 + t q_2^2 + 18\alpha^3 - 6\alpha^2 q_1 - 6\alpha^2 q_2 + 2\alpha q_1 q_2}{18(t^2-\alpha^2)} \end{cases} \tag{4.20}$$

Proof. See Appendix 4.8.6. □

The novelty of this section emerges from expressions (4.18), (4.19) and (4.20). Note that if $q_1 = q_2 \equiv q$ we obtain the same equilibrium outcomes of the non-discriminatory regime. However, now platforms price discriminate between the two sides based on the distinct perceived degree of vertical differentiation by the platforms' members. Indeed, if $q_1 \geq q_2$, then $p_1^{A*} \leq p_2^{A*}$ and $p_1^{B*} \geq p_2^{B*}$.

In light of (4.18), we obtain that platform 2 may use a divide and conquer strategy- Under Assumption 4.3, this is the case if

$$q_1 > 3\,(t - \alpha)\,.$$

This means that we may have a divide and conquer strategy conducted by the low-quality platform in equilibrium. The high-quality platform is relatively more efficient on side 1 for $q_1 > q_2$. Thus, the low-quality platform needs to decrease the price charged to side 1 in order to conquer an extra market share.

By providing a subsidy to the side where the high-quality platform has a greater quality advantage, the low-quality platform conquers additional valuable (side 1) agents of the high-quality platform and, by the presence of network effects it also attracts consumers from the opposite side. Thus, the adoption of such strategy counterbalances the negative effect of being the active platform whose quality is lower.

As the degree of vertical differentiation (q_j) increases, the equilibrium price of the high-quality platform increases while the equilibrium price of the low quality platform decreases in the corresponding side of the market. Both prices vary in opposite ways but with the same magnitude:

$$\frac{\partial p_j^{A*}}{\partial q_j} < 0 < \frac{\partial p_j^{B*}}{\partial q_j} \text{ and } \frac{\partial p_j^{B*}}{\partial q_j} = \left| \frac{\partial p_j^{A*}}{\partial q_j} \right|.$$

On each side of the market, as the degree of vertical differentiation increases, the equilibrium market share of the high-quality platform increases while the equilibrium market share of the low-quality platform decreases. With the incorporation of discrimination on the perceived quality between the two sides of the market, for $q_1 \geq q_2$, the equilibrium market share of the high-quality platform on side 1 is higher than the equilibrium market share of the high-quality platform on side 2 (because $t > \alpha$[6]).

As the quality gaps between platforms (q_j) increase, the correspondent equilibrium profit π^{B*} $\left(\pi^{A*}\right)$ increases (decreases) since $\frac{\partial \pi^{A*}}{\partial q_j} < 0$ and $\frac{\partial \pi^{B*}}{\partial q_j} > 0$ under Assumption 4.3 (see Appendix 4.8.6).

4.5.2 Impacts of discrimination

To better understand the consequences of the introduction of a discriminatory regime, suppose $q_1 = q + \varepsilon$ and $q_2 = q - \varepsilon$, with $q_1 \geq 0$, $q_2 \geq 0$ and $\varepsilon \in [0, q]$.

Notice that $q_1 - q_2 = 2\varepsilon$, which can be interpreted as the asymmetry on the degree of quality between the two sides of the market.

In this case, the degree of vertical differentiation between the platforms is higher for side 1. Thus, although both sides consider platform B as the high-quality platform, side 1 assigns a higher quality advantage to platform B relatively to platform A in comparison with side 2.

If $\varepsilon = 0$ we fall in the benchmark case where $q_1 = q_2 \equiv q$.

Proposition 19. ***(Impacts of discrimination)***

As the asymmetry of the quality gap between the sides increases $(\uparrow \varepsilon)$*:*

(i) *the equilibrium price of side 1 decreases in the low-quality platform and increases in the high-quality platform* $\left(\downarrow p_1^{A*}, \uparrow p_1^{B*}\right)$ *and opposite occurs for the equilibrium price of side*

[6]The same argument stands for platform A but with an opposite effect relatively to the degree of vertical differentiation.

2 $\left(\uparrow p_2^{A*}, \downarrow p_2^{B*}\right)$;

(ii) the equilibrium market share of side 1 decreases on the low-quality platform and increases on the high-quality platform $\left(\downarrow D_1^{A*}, \uparrow D_1^{B*}\right)$ *and the equilibrium market share of side 2 decreases on the high-quality platform and increases on the low-quality platform* $\left(\downarrow D_2^{B*}, \uparrow D_2^{A*}\right)$;

(iii) the equilibrium profits of both platforms increase $\left(\uparrow \pi^{A*}, \uparrow \pi^{B*}\right)$.

Proof. See Appendix 4.8.6. □

Proposition 19 establishes that in equilibrium a greater quality gap of qualities between the two sides of the market sides is profit enhancing for both platforms.

4.6 GROUP DISCRIMINATION AND ENDOGENOUS QUALITY

Assume that the platforms' quality can be different between the two sides of the market. Also, for simplicity, let $q_1 \equiv V_1^B - V_1^A$ and $q_2 \equiv V_2^B - V_2^A$. Consider now an investment stage, where only the quality of the high-quality platform is endogenously determined on both sides of the market. Then, assume that $\beta_j^A = +\infty$ such that the low-quality platform does not invest in quality, implying that $V_j^A = 0$, $j = 1, 2$.

Then, platform B has to invest $I\left(V_j^B\right)$ to obtain a quality level V_j^B. We assume that $I\left(V_j^B\right) = \frac{1}{2}\beta_j^B {V^B}^2$. The parameter β_j^B measures the cost of quality provision of the high-quality platform on side j, $j = 1, 2$.

If the cost of quality provision parameters are sufficiently high, platform B will always choose quality levels that satisfy Assumption 4.3. Then, using (4.20), and also considering the cost of quality implies the following maximization problem for the platform B:

$$max_{V_1^B, V_2^B} \frac{18t^3 - 18t^2\alpha + 6t^2 V_1^B + 6t^2 V_2^B - 18t\alpha^2 + t\left(V_1^B\right)^2 + t\left(V_2^B\right)^2 + 18\alpha^3 - 6\alpha^2 V_1^B - 6\alpha^2 V_2^B + 2\alpha V_1^B V_2^B}{18(t^2 - \alpha^2)} - \frac{\sum_{j=1}^{2} \beta_j^B \left(V_j^B\right)^2}{2}. \tag{4.21}$$

The first order conditions imply:

$$\begin{aligned}\frac{\partial \pi^{B*}}{\partial V_1^B} &= 0 \Leftrightarrow \frac{6t^2+2tV_1^B-6\alpha^2+2\alpha V_2^B}{18(t^2-\alpha^2)} - \beta_1^B V_1^B = 0; \qquad (4.22)\\ \frac{\partial \pi^{B*}}{\partial V_2^B} &= 0 \Leftrightarrow \frac{6t^2+2tV_2^B-6\alpha^2+2\alpha V_1^B}{18(t^2-\alpha^2)} - \beta_2^B V_2^B = 0. \qquad (4.23)\end{aligned}$$

Solving the system (4.22)-(4.23), we obtain the equilibrium qualities of platform B on both sides of the market:

$$\begin{aligned}V_1^{B*} &= \frac{3(t-\alpha)\left[9\beta_2^B(t+\alpha)-1\right]}{9\left[9\beta_1^B\beta_2^B(t-\alpha)(t+\alpha)-\left(\beta_1^B+\beta_2^B\right)t\right]+1}; \qquad (4.24)\\ V_2^{B*} &= \frac{3(t-\alpha)\left[9\beta_1^B(t+\alpha)-1\right]}{9\left[9\beta_1^B\beta_2^B(t-\alpha)(t+\alpha)-\left(\beta_1^B+\beta_2^B\right)t\right]+1}. \qquad (4.25)\end{aligned}$$

Solving the above expressions for β_1^B and β_2^B, we obtain:

$$\beta_1 = \frac{tq_1 + \alpha q_2 + 3\left(t^2-\alpha^2\right)}{9q_1\left(t^2-\alpha^2\right)}, \qquad (4.26)$$

$$\beta_2 = \frac{tq_2 + \alpha q_1 + 3\left(t^2-\alpha^2\right)}{9q_2\left(t^2-\alpha^2\right)}. \qquad (4.27)$$

This means that working with q_1 and q_2 instead of β_1^B and β_2^B merely corresponds to changing variables. In particular, the previous results obtained for $q_1 > q_2$ remain valid if we impose $\beta_1^B < \beta_2^B$.

4.7 CONCLUSIONS

In this manuscript, we investigate the simultaneous incorporation of vertical and horizontal differentiation in two-sided markets. We prove that as the degree of vertical differentiation increases the equilibrium prices, market shares and profits of the low-quality (high-quality) platform decrease (increase).

As an extension, we allow platforms to previously choose their quality level by introducing a two-stage game where platforms first choose their quality levels and then compete on prices. We conclude that with pure singlehoming consumers, the platform that gets higher profits is the one whose investment in quality is higher.

Finally, we propose a generalized version of our benchmark model in order to allow for asymmetries between the two sides of the market. We conclude that this situation is profit enhancing for both platforms.

Moreover, divide and conquer strategies appear in equilibrium when we consider that only the high-quality platform invests in quality. Indeed, the low-quality platform may charge a negative access price on the side where the high-quality platform has a lower cost of quality production. By providing a subsidy to this side, the low-quality platform conquers additional agents of this side, which allows it to attract more consumers from the opposite side.

In our future research, we would like to incorporate (partial) multihoming and the role of incomplete information between sides. The incorporation of endogenous locations with quadratic marginal costs relatively to distance in a model with three stages (in stage one platforms would decide locations, on stage two platforms would decide quality levels and stage three would correspond to pricing decision) is another possible extension.

4.8 APPENDIX

4.8.1 Demand and profits

Proof of Proposition 9

Indifferent consumer and demand functions

The four consumer utilities assuming the existence of platforms $i \in \{A, B\}$ and sides $j \in \{1, 2\}$ are given by:

$$\begin{cases} u_1^A(x_1^A) = V^A + \alpha D_2^A - p_1^A - t x_1^A; \\ u_1^B(x_1^A) = V^B + \alpha D_2^B - p_1^B - t(1 - x_1^A); \\ u_2^A(x_2^A) = V^A + \alpha D_1^A - p_2^A - t x_2^A; \\ u_2^B(x_2^A) = V^B + \alpha D_1^B - p_2^B - t(1 - x_2^A). \end{cases}$$

The indifferent consumer on both sides of the market follows by setting:

$$\begin{aligned} u_1^A(x_1) = u_1^B(x_1) \Leftrightarrow V^A + \alpha D_2^A - p_1^A - t x_1^A &= V^B + \alpha D_2^B - p_1^B - t(1 - x_1^A) \\ \Leftrightarrow x_1^A &= \tfrac{1}{2} + \tfrac{V^A - V^B + \alpha\left(2D_2^A - 1\right) - \left(p_1^A - p_1^B\right)}{2t}, \end{aligned}$$

and also:

$$\begin{aligned} u_2^A(x_2) = u_2^B(x_2) \Leftrightarrow V^A + \alpha D_1^A - p_2^A - t x_2^A &= V^B + \alpha D_1^B - p_2^B - t(1 - x_2^A) \\ \Leftrightarrow x_2^A &= \tfrac{1}{2} + \tfrac{V^A - V^B + \alpha\left(2D_1^A - 1\right) - \left(p_2^A - p_2^B\right)}{2t}. \end{aligned}$$

The demands as a function of prices and platforms' qualities are given by:

$$\begin{aligned} D_1^A &= \tfrac{1}{2} + \tfrac{V^A - V^B + \alpha\left(2D_2^A - 1\right) - \left(p_1^A - p_1^B\right)}{2t} \\ \Leftrightarrow D_1^A &= \tfrac{1}{2} + \frac{V^A - V^B + \alpha\left\{2\left[\frac{1}{2} + \frac{\left(V^A - V^B\right) + \alpha\left(2D_1^A - 1\right) - \left(p_2^A - p_2^B\right)}{2t}\right] - 1\right\} - \left(p_1^A - p_1^B\right)}{2t}, \end{aligned}$$

from which follows:

$$D_1^A\left(p_j^i\right) = \frac{1}{2} + \frac{\alpha\left(p_2^B - p_2^A\right) + t\left(p_1^B - p_1^A\right)}{2\left(t - \alpha\right)\left(t + \alpha\right)} + \frac{V^A - V^B}{2\left(t - \alpha\right)}.$$

By symmetry, we obtain:

$$D_2^A\left(p_j^i\right) = \tfrac{1}{2} + \tfrac{\alpha\left(p_1^B - p_1^A\right) + t\left(p_2^B - p_2^A\right)}{2(t-\alpha)(t+\alpha)} + \tfrac{V^A - V^B}{2(t-\alpha)}.$$

Since total inelastic demand equals 1, it follows that:

$$D_j^B\left(p_j^i\right) = 1 - D_j^A\left(p_j^i\right).$$

Reaction functions

The profit functions are given by:

$$\begin{cases} \pi^A(P) = p_1^A D_1^A(P) + p_2^A D_2^A(P); \\ \pi^B(P) = p_1^B D_1^B(P) + p_2^B D_2^B(P), \end{cases}$$

where $P \equiv \left(p_1^A, p_2^A, p_1^B, p_2^A\right)$ Then, the reaction functions are given by:

$$\begin{cases} \frac{\partial \pi^A}{\partial p_1^A} = 0 \\ \frac{\partial \pi^A}{\partial p_2^A} = 0 \\ \frac{\partial \pi^B}{\partial p_1^B} = 0 \\ \frac{\partial \pi^B}{\partial p_2^B} = 0 \end{cases} \Leftrightarrow \begin{cases} -2tp_1^A - 2\alpha p_2^A + tp_1^B + \alpha p_2^B = -\left(t^2-\alpha^2\right) - (V^A - V^B)(t+\alpha); \\ -2\alpha p_1^A - 2tp_2^A + \alpha p_1^B + tp_2^B = -\left(t^2-\alpha^2\right) - (V^A - V^B)(t+\alpha); \\ tp_1^A + \alpha p_2^A - 2tp_1^B - 2\alpha p_2^B = -\left(t^2-\alpha^2\right) - (V^B - V^A)(t+\alpha); \\ \alpha p_1^A + tp_2^A - 2\alpha p_1^B - 2tp_2^B = -\left(t^2-\alpha^2\right) - (V^B - V^A)(t+\alpha). \end{cases}$$

From the solution of this system of equations yields the corresponding equilibrium prices from expression (4.4), the equilibrium market shares of expression (4.5) and the equilibrium profits of expression (4.6).

Global profit maximization (Assumption 4.1)

The second order conditions require:

$$|H_1| = \frac{\partial^2 \pi^A}{\partial {p_1^A}^2} = \frac{-2t}{2(t^2-\alpha^2)} = -\frac{t}{(t-\alpha)(t+\alpha)} < 0.$$

Computing the second order cross-partial derivative we obtain:

$$\frac{\partial^2 \pi^A}{\partial p_1^A \partial p_2^A} = \frac{-2\alpha}{2(t^2-\alpha^2)} = -\frac{\alpha}{(t-\alpha)(t+\alpha)}.$$

Also note that:

$$\frac{\partial^2 \pi^A}{\partial {p_2^A}^2} = -\frac{t}{(t-\alpha)(t+\alpha)}$$

To accomplish global profit maximization is required:

$$|H_2| > 0 \Leftrightarrow \frac{t^2}{[(t-\alpha)(t+\alpha)]^2} > \frac{\alpha^2}{[(t-\alpha)(t+\alpha)]^2} \Leftrightarrow t > \alpha.$$

which holds under Assumption 4.1.

Positive prices and market shares (Assumption 4.2)

We now check that $x^* \in [0,1]$. Substituting (4.4) and (4.5) into (4.3) yields:

$$x^* = \frac{1}{2} - \frac{q}{6(t-\alpha)}. \tag{4.28}$$

Observe that $x^* < 1$ because $q \geq 0$.

The accomplishment of $x^* > 0$ requires:

$$\frac{1}{2} - \frac{V^B - V^A}{6(t-\alpha)} > 0 \Leftrightarrow V^B - V^A < 3(t-\alpha) \Leftrightarrow q < 3(t-\alpha),$$

which coincides with Assumption 4.2. $\square$

4.8.2 Welfare analysis

Proof of Lemma 11

We start by computing the consumer surplus, which is given by:

$$CS = \int_0^{\widetilde{x}} u^A dx + \int_{\widetilde{x}}^1 u^B dx.$$

Computing we obtain:

$$\Leftrightarrow CS = \left(V^A \left[x\right]_0^{\widetilde{x}} + \alpha D^A \left[x\right]_0^{\widetilde{x}} - p^A \left[x\right]_0^{\widetilde{x}} - t\left[\tfrac{x^2}{2}\right]_0^{\widetilde{x}} \right) + \left(V^B \left[x\right]_{\widetilde{x}}^1 + \alpha D^B \left[x\right]_{\widetilde{x}}^1 - p^B \left[x\right]_{\widetilde{x}}^1 - t\left[x\right]_{\widetilde{x}}^1 + t\left[\tfrac{x^2}{2}\right]_{\widetilde{x}}^1 \right)$$

$$\Leftrightarrow CS = V^A \widetilde{x} + \alpha D^A \widetilde{x} - p^A \widetilde{x} - t\tfrac{\widetilde{x}^2}{2} + V^B\left(1-\widetilde{x}\right) + \alpha D^B\left(1-\widetilde{x}\right) - p^B\left(1-\widetilde{x}\right) - t\left(1-\widetilde{x}\right) + t\left(\tfrac{1}{2} - \tfrac{\widetilde{x}^2}{2}\right)$$

Rearranging, it follows:

$$CS = V^B - \widetilde{x}(V^B - V^A) + \alpha D^B - \alpha\widetilde{x}(D^B - D^A) - p^B + \widetilde{x}(p^B - p^A) - \tfrac{t}{2} + t\widetilde{x}(1-\widetilde{x}) \tag{4.29}$$

Substituting expressions (4.4) and (4.5) in expression (4.29) and taking into account that $q \equiv V^B - V^A$ yields:

$$\begin{aligned} CS^* \;=\; & V^B - \left[\tfrac{1}{2} - \tfrac{q}{6(t-\alpha)}\right] q + \alpha \left[\frac{1}{2} + \tfrac{q}{6(t-\alpha)}\right] - \\ & \alpha \left[\tfrac{1}{2} - \tfrac{q}{6(t-\alpha)}\right] \left\{ \left[\tfrac{1}{2} + \tfrac{q}{6(t-\alpha)}\right] - \left[\tfrac{1}{2} - \tfrac{q}{6(t-\alpha)}\right] \right\} - \\ & \left(\tfrac{q}{3} + t - \alpha\right) + \left[\tfrac{1}{2} - \tfrac{q}{6(t-\alpha)}\right] \left[\left(\tfrac{q}{3} + t - \alpha\right) - \left(-\tfrac{q}{3} + t - \alpha\right)\right] \\ & -\tfrac{t}{2} + t \left[\tfrac{1}{2} - \tfrac{q}{6(t-\alpha)}\right] \left\{1 - \left[\tfrac{1}{2} - \tfrac{q}{6(t-\alpha)}\right]\right\} \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \;\; & CS^* = V^B - \frac{q}{2} + \tfrac{q^2}{6(t-\alpha)} + \frac{\alpha}{2} + \tfrac{\alpha q}{6(t-\alpha)} - \alpha \left[\tfrac{q}{6(t-\alpha)} - \tfrac{q^2}{18(t-\alpha)^2}\right] - \\ & \tfrac{q}{3} - t + \alpha + \tfrac{q}{3} - \tfrac{q^2}{9(t-\alpha)} - \tfrac{t}{2} + t \left[\frac{1}{4} - \tfrac{q^2}{36(t-\alpha)^2}\right] \end{aligned}$$

Rearranging we obtain:

$$\begin{aligned} \Leftrightarrow \;\; & CS^* = \tfrac{V^B}{2} + \tfrac{V^A}{2} + \frac{3}{2}\alpha - \tfrac{5}{4}t + \tfrac{q^2}{18(t-\alpha)} - \tfrac{q^2}{36(t-\alpha)^2}(t - 2\alpha) \\ \Leftrightarrow \;\; & CS^* = \tfrac{V^B + V^A + 3\alpha}{2} - \tfrac{5}{4}t + \tfrac{q^2}{36(t-\alpha)^2}t, \end{aligned}$$

which coincides with expression (4.7) since $q = V^B - V^A$.

Differentiating the consumer surplus relatively to V^B and V^A we get:

$$\frac{\partial CS^*}{\partial V^A} = \frac{1}{2} - \frac{t}{18(t-\alpha)^2} q \gtrless 0; \tag{4.30}$$

$$\frac{\partial CS^*}{\partial V^B} = \frac{1}{2} + \frac{t}{18(t-\alpha)^2} q > 0. \tag{4.31}$$

Expression (4.30) is positive if and only if:

$$q < \tfrac{9(t-\alpha)^2}{t}, \tag{4.32}$$

which is exactly expression (4.8). Then, Lemma 11 is straightforward. □

Proof of Lemma 12

The total welfare equals:

$$W = \int_0^{\widetilde{x}} u^A dx + \int_{\widetilde{x}}^1 u^B dx,$$

it follows that:

$$\Leftrightarrow W = \left(V^A\left[x\right]_0^{\widetilde{x}} + \alpha D^A\left[x\right]_0^{\widetilde{x}} - t\left[\frac{x^2}{2}\right]_0^{\widetilde{x}}\right) + \left(V^B\left[x\right]_{\widetilde{x}}^1 + \alpha D^B\left[x\right]_{\widetilde{x}}^1 - t\left[x\right]_{\widetilde{x}}^1 + t\left[\frac{x^2}{2}\right]_{\widetilde{x}}^1\right)$$

$$\Leftrightarrow W = V^A\widetilde{x} + \alpha D^A\widetilde{x} - t\frac{\widetilde{x}^2}{2} + V^B\left(1-\widetilde{x}\right) + \alpha D^B\left(1-\widetilde{x}\right) - t\left(1-\widetilde{x}\right) + t\left(\frac{1}{2} - \frac{\widetilde{x}^2}{2}\right)$$

$$\Leftrightarrow W = V^A\widetilde{x} + \alpha D^A\widetilde{x} - t\frac{\widetilde{x}^2}{2} + V^B\left(1-\widetilde{x}\right) + \alpha D^B\left(1-\widetilde{x}\right) - t\left(1-\widetilde{x}\right) + t\left(\frac{1}{2} - \frac{\widetilde{x}^2}{2}\right),$$

from which results:

$$W = V^B - \widetilde{x}(V^B - V^A) + \alpha D^B - \alpha\widetilde{x}(D^B - D^A) - \frac{t}{2} + t\widetilde{x}(1-\widetilde{x}). \tag{4.33}$$

Substituting expressions (4.4) and (4.5) in expression (4.33) yields:

$$\begin{aligned} W^* \;\; = \;\; & V^B - \left[\tfrac{1}{2} - \tfrac{V^B - V^A}{6(t-\alpha)}\right](V^B - V^A) + \alpha\left[\tfrac{1}{2} + \tfrac{V^B - V^A}{6(t-\alpha)}\right] - \\ & \alpha\left[\tfrac{1}{2} - \tfrac{V^B - V^A}{6(t-\alpha)}\right]\left\{\left[\tfrac{1}{2} + \tfrac{V^B - V^A}{6(t-\alpha)}\right] - \left[\tfrac{1}{2} - \tfrac{V^B - V^A}{6(t-\alpha)}\right]\right\} - \\ & \tfrac{t}{2} + t\left[\tfrac{1}{2} - \tfrac{V^B - V^A}{6(t-\alpha)}\right]\left\{1 - \left[\tfrac{1}{2} - \tfrac{V^B - V^A}{6(t-\alpha)}\right]\right\} \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \;\; & W^* = V^B - \tfrac{1}{2}(V^B - V^A) + \tfrac{\left(V^B - V^A\right)^2}{6(t-\alpha)} + \tfrac{\alpha}{2} + \tfrac{\alpha\left(V^B - V^A\right)}{6(t-\alpha)} - \\ & \alpha\left[\tfrac{V^B - V^A}{6(t-\alpha)} - \tfrac{\left(V^B - V^A\right)^2}{18(t-\alpha)^2}\right] - \tfrac{t}{2} + t\left[\tfrac{1}{4} - \tfrac{\left(V^B - V^A\right)^2}{36(t-\alpha)^2}\right] \end{aligned}$$

$$\Leftrightarrow W^* = \tfrac{V^B + V^A}{2} + \tfrac{1}{2}\alpha - \tfrac{1}{4}t + \tfrac{\left(V^B - V^A\right)^2}{36(t-\alpha)^2}(2\alpha - t + 6t - 6\alpha)$$

resulting:

$$W^* = \tfrac{V^B + V^A + \alpha}{2} - \tfrac{1}{4}t + \tfrac{(V^B - V^A)^2}{36(t-\alpha)^2}(5t - 4\alpha).$$

Multiplying by 2 (the number of sides of the market), we obtain expression (4.9) since $q = V^B - V^A$.[7]

Differentiating the total welfare relatively to V^B and V^A we obtain:

$$\frac{\partial W}{\partial V^A} = 1 - \frac{t + 4(t-\alpha)}{9(t-\alpha)^2} q \gtrless 0; \tag{4.34}$$

[7] Alternatively, we could compute $W^* = 2CS_j^* + \pi^{A*} + \pi^{B*}$ to obtain a similar result.

$$\frac{\partial W}{\partial V^B} = 1 + \frac{t + 4(t-\alpha)}{9(t-\alpha)^2} q > 0. \tag{4.35}$$

Expression (4.34) is positive if and only if:

$$q < \tfrac{9(t-\alpha)^2}{t+4(t-\alpha)}, \tag{4.36}$$

which is exactly expression (4.10). Then, Lemma 12 is straightforward. □

4.8.3 Tipping

Proof of Lemma 13

Tipping in favor of the high-quality platform B

Tipping in favor of the high-quality platform corresponds to a market situation where:

$$\begin{cases} \widetilde{x}_1 = \widetilde{x}_2 = 0; \\ p_1^{A*} = p_2^{A*} = 0. \end{cases} \tag{4.37}$$

Substituting (4.37) into (4.3) on both sides of the market yields the following system of two linear equations:

$$p_1^B = \frac{q(t+\alpha)}{t} - \frac{(t-\alpha)(t+\alpha)}{t} - \frac{\alpha}{t} p_2^B; \tag{4.38}$$

$$p_2^B = \frac{q(t+\alpha)}{t} - \frac{(t-\alpha)(t+\alpha)}{t} - \frac{\alpha}{t} p_1^B. \tag{4.39}$$

Solving (4.38) and (4.39) simultaneously, results the tipping prices:

$$p_1^{B(T)*} = p_2^{B(T)*} = \alpha - (t-q).$$

Thus, the equilibrium profit of platform B under tipping equals:

$$\pi^{B(T)*} = 2\alpha - 2(t-q).$$

Tipping in favor of the low-quality platform A

Tipping in favor of the low-quality platform corresponds to a market situation where:

$$\begin{cases} \widetilde{x}_1 = \widetilde{x}_2 = 1; \\ p_1^{B*} = p_2^{B*} = 0. \end{cases} \tag{4.40}$$

Substituting (4.40) into (4.3) on both sides of the market yields the following system of two linear equations:

$$p_1^A = -\frac{q(t+\alpha)}{t} - \frac{(t-\alpha)(t+\alpha)}{t} - \frac{\alpha}{t}p_2^A; \tag{4.41}$$

$$p_2^A = -\frac{q(t+\alpha)}{t} - \frac{(t-\alpha)(t+\alpha)}{t} - \frac{\alpha}{t}p_1^A. \tag{4.42}$$

Solving (4.41) and (4.42) simultaneously, results the candidate tipping prices:

$$p_1^{A(T)*} = p_2^{A(T)*} = \alpha - (t+q).$$

Since these prices are negative, tipping in favor of platform A cannot occur.

Proof of Proposition 14

The tipping profit of the high-quality platform is higher than the profit under competition if and only if:

$$\pi^{B(T)*} \geq \pi^{B(Interior)*} \Leftrightarrow 2\alpha - 2(t-q) \geq \tfrac{[3(t-\alpha)+q]^2}{9(t-\alpha)}.$$

This condition is equivalent to non-negativity of the following polynomial:

$$-27\alpha^2 + (54t - 12q)\alpha + (12qt - q^2 - 27t^2) \geq 0 \tag{4.43}$$

The roots of the polynomial are given by:

$$\alpha_{C1} = t - \frac{q}{3} \quad \cap \quad \alpha_{C2} = t - \frac{q}{9}.$$

By Assumption 4.2, the interior equilibrium profit $\pi^{B(Interior)*}$ only holds for $\alpha < t - \frac{q}{3}$. By the sign of (4.43) follows that for all $\alpha < t - \frac{q}{3}$ yields $\pi^{B(T)*} < \pi^{B(Interior)*}$ and, therefore, the tipping profit $\pi^{B(T)*}$ is preferable only for $\alpha > t - \frac{q}{3}$. $\square$

4.8.4 Social optimum

Proof of Proposition 15

From (4.33), the exact value of the total welfare as a function of $\widetilde{x}$ is given by:

$$\begin{aligned} W^* &= V^B - \widetilde{x}(V^B - V^A) + \alpha D^B - \alpha\widetilde{x}(D^B - D^A) - \tfrac{t}{2} + t\widetilde{x}(1-\widetilde{x}) \Leftrightarrow \\ W^* &= V^B - \widetilde{x}(V^B - V^A) + \alpha(1-\widetilde{x}) - \alpha\widetilde{x}(1-2\widetilde{x}) - \tfrac{t}{2} + t\widetilde{x}(1-\widetilde{x}). \end{aligned}$$

Differentiating (4.33) relatively to $\widetilde{x}$ yields expression (4.12).

(i) Then, we determine the maximizer:

$$\begin{aligned} \tfrac{\partial W(\widetilde{x})}{\partial \widetilde{x}} = 0 \quad &\Leftrightarrow -\left(V^B - V^A\right) - 2\alpha + 4\alpha\widetilde{x} + t - 2t\widetilde{x} = 0 \\ &\Leftrightarrow \tilde{x}^{FB} = \tfrac{1}{2} - \tfrac{\left(V^B - V^A\right)}{2}\left(\tfrac{1}{t-2\alpha}\right), \end{aligned}$$

which is expression (4.12) since $q = V^B - V^A$.

(ii) The second order condition is given by:

$$\tfrac{\partial^2 W(\widetilde{x})}{\partial \widetilde{x}^2} = 4\alpha - 2t,$$

which is negative if $\alpha < \frac{t}{2}$ and, thus, expression (4.12) is a global maximizer if and only if:

$$\alpha < \frac{t}{2}. \tag{4.44}$$

(iii) Then, we check whether $0 \leq \tilde{x}^{FB} \leq \frac{1}{2}$. First:

$$\tilde{x}^{FB} \leq \frac{1}{2} \Leftrightarrow \frac{V^B - V^A}{2}\left(\frac{1}{t-2\alpha}\right) \geq 0.$$

Then:

$$\tilde{x}^{FB} \geq 0 \Leftrightarrow V^B - V^A \leq t - 2\alpha.$$

(iv) Finally, we check whether $\tilde{x}^{FB} < \tilde{x}^*$. We know from (4.28) that:

$$\tilde{x}^* = \frac{1}{2} - \frac{V^B - V^A}{6(t-\alpha)}.$$

Thus, $\tilde{x}^{FB} < \tilde{x}^*$ implies:

$$\begin{aligned}
\tfrac{1}{2} - \tfrac{V^B - V^A}{2}\left(\tfrac{1}{t-2\alpha}\right) < \tfrac{1}{2} - \tfrac{V^B - V^A}{6(t-\alpha)} \Leftrightarrow -\tfrac{V^B - V^A}{2}\left(\tfrac{1}{t-2\alpha}\right) &< -\tfrac{V^B - V^A}{6(t-\alpha)} \\
\Leftrightarrow \left(V^A - V^B\right)(3t - 3\alpha) &< \left(V^A - V^B\right)(t - 2\alpha) \\
\Leftrightarrow (V^A - V^B)(2t - \alpha) &< 0 \\
\Leftrightarrow -q(2t - \alpha) &< 0
\end{aligned}$$

which always holds under expression (4.44). Thus, it follows:

$$0 \leq \tilde{x}^{FB} < \tilde{x}^* \leq \frac{1}{2}.$$

(v) For $\frac{t}{2} < \alpha < t$, $\widetilde{x}^{FB}$ is not a global maximizer. In this case, the social optimum is attained at one of the two extremes of the linear city: $\widetilde{x}^{FB} = 0$ or $\widetilde{x}^{FB} = 1$. Thus, the regulator intends to concentrate all the agents of the market in the platform whose quality is higher: platform B. □

4.8.5 Endogenous quality

The last stage remains unchanged from which it follows that expressions (4.4) and (4.5) are second stage equilibrium outcomes.

Proof of Proposition 16

Interior equilibrium qualities

Each platform, $i \in \{A, B\}$, invests in quality and solves the following maximization problem:

$$\max_{V^i} \pi^{i*} \;\Leftrightarrow\; \max_{V^i} \tfrac{\left[3(t-\alpha)+V^i - V^{-i}\right]^2}{9(t-\alpha)} - \tfrac{1}{2}\beta_i {V^i}^2. \tag{4.45}$$

The first-order conditions of the maximization problems are:

$$\frac{\partial \pi^{B*}}{\partial V^B} = 0 \;\Leftrightarrow\; V^B = \tfrac{6(t-\alpha)-2V^A}{9(t-\alpha)\beta_B - 2}; \tag{4.46}$$

$$\frac{\partial \pi^{A*}}{\partial V^A} = 0 \;\Leftrightarrow\; V^A = \tfrac{6(t-\alpha)-2V^B}{9(t-\alpha)\beta_A - 2}. \tag{4.47}$$

Solving the system (4.46)-(4.47), we obtain the equilibrium qualities:

$$V^{B*} = \frac{2}{3}\left\{\frac{9(t-\alpha)\beta_A - 4}{\beta_B[9(1-\alpha)\beta_A - 2] - 2\beta_A}\right\}; \tag{4.48}$$

$$V^{A*} = \frac{2}{3}\left\{\frac{9(t-\alpha)\beta_B - 4}{\beta_B[9(1-\alpha)\beta_A - 2] - 2\beta_A}\right\}. \tag{4.49}$$

Subtracting (4.49) from (4.48) yields the following quality difference between the platforms:

$$q^* \equiv V^{B*}(\beta_B, \beta_A) - V^{A*}(\beta_B, \beta_A) = \frac{6(t-\alpha)(\beta_A - \beta_B)}{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A},$$

which is exactly expression (4.13).

We now show that the denominator of (4.13) is positive. It is given by:

$$\beta_B\,[9(1-\alpha)\beta_A - 2] - 2\beta_A.$$

Since $\beta_A > \frac{2}{9(t-\alpha)}$, it is increasing in β_B. Replacing β_B by its lower bound, $\frac{4}{9(t-\alpha)}$, we obtain a sufficient condition for the denominator to be positive:

$$\frac{4}{9(t-\alpha)}\,[9(1-\alpha)\beta_A - 2] - 2\beta_A > 0 \;\Leftrightarrow\; 2\beta_A - \frac{8}{9(t-\alpha)} > 0,$$

which is always true under the hypothesis of the Proposition.

The numerators of (4.48) and (4.49) are also positive, because we are working under the assumption that $\beta_A \geq \beta_B > \frac{4}{9(t-\alpha)}$. This allows us to conclude that $V^{B*} \geq V^{A*} > 0$.

We must also verify that Assumption 4.2 is satisfied in equilibrium:

$$q^* \leq 3(t-\alpha).$$

From expression (4.13), Assumption 4.2 is verified if and only if:

$$\frac{6(t-\alpha)(\beta_A - \beta_B)}{\beta_B[9(t-\alpha)\beta_A - 2\,] - 2\beta_A} \leq 3(t-\alpha) \;\Leftrightarrow\; \beta_B > \frac{4}{9(t-\alpha)}, \tag{4.50}$$

which is always true under the hypothesis of the Proposition.

Interior equilibrium prices, market shares and profits

Substituting (4.13) in expressions (4.4), (4.5) and (4.45), we obtain:

$$\begin{cases} p^{A*} = \beta_A \left\{ \frac{(t-\alpha)[9(t-\alpha)\beta_B - 4]}{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A} \right\}; \quad p^{B*} = \beta_B \left\{ \frac{(t-\alpha)[9(t-\alpha)\beta_A - 4]}{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A} \right\}; \\ \\ D_j^{A*} = \frac{1}{2} - \frac{\beta_A - \beta_B}{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A}; \quad D_j^{B*} = \frac{1}{2} + \frac{\beta_A - \beta_B}{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A}; \\ \\ \pi^{A*} = \frac{\beta_A[9(t-\alpha)\beta_B-4]^2[9(t-\alpha)\beta_A-2]}{9\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}; \quad \pi^{B*} = \frac{\beta_B[9(t-\alpha)\beta_A-4]^2[9(t-\alpha)\beta_B-2]}{9\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}. \end{cases}$$

which correspond to expressions (4.14), (4.15) and (4.16), respectively.

The second derivatives with respect to qualities are given by:

$$\begin{cases} \frac{\partial^2 \pi^{B*}}{\partial {V^B}^2} = \frac{2-9(t-\alpha)\beta_B}{9(t-\alpha)}; \\ \frac{\partial^2 \pi^{A*}}{\partial {V^A}^2} = \frac{2-9(t-\alpha)\beta_A}{9(t-\alpha)}. \end{cases}$$

Since $\beta_A \geq \beta_B > \frac{4}{9(t-\alpha)}$, the second-derivatives are strictly negative, which means that the second-order conditions are satisfied.

Market shares in the unit segment

Finally, we verify that $D^{A*} \geq 0$ (and $D^{B*} \leq 1$):

$$\begin{aligned} D^{A*} \geq 0 \; &\Leftrightarrow \; \frac{\beta_A - \beta_B}{\beta_B\left[9(t-\alpha)\beta_A - 2\right] - 2\beta_A} \leq \frac{1}{2} \\ &\Leftrightarrow \; 2\beta_A - 2\beta_B \leq \beta_B\left[9(t-\alpha)\beta_A - 2\right] - 2\beta_A \\ &\Leftrightarrow \; \beta_B \geq \frac{4}{9(t-\alpha)}, \end{aligned}$$

which is true. □

Proof of Lemma 17

Differentiating expressions (4.14), (4.15) and (4.16) with respect to β_B and $\beta_{A.}$ implies the results and respective impacts described in the table below.

Note also that: $\frac{\partial p^{B*}}{\partial \beta_A} = \left|\frac{\partial p^{A*}}{\partial \beta_A}\right|$; $\frac{\partial p^{A*}}{\partial \beta_B} = \left|\frac{\partial p^{B*}}{\partial \beta_B}\right|$ and $\frac{\partial D_j^{B*}}{\partial \beta_A} = \left|\frac{\partial D_j^{A*}}{\partial \beta_A}\right|$; $\frac{\partial D_j^{A*}}{\partial \beta_B} = \left|\frac{\partial D_j^{B*}}{\partial \beta_B}\right|$.

Platform A	**Platform B**
$\frac{\partial q^*}{\partial \beta_A} = \frac{6(t-\alpha)\beta_B[9(1-\alpha)\beta_B-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(>0)$	$\frac{\partial q^*}{\partial \beta_B} = -\frac{6(t-\alpha)\beta_A[9(1-\alpha)\beta_B-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(<0)$
$\frac{\partial p^{A*}}{\partial \beta_A} = -\frac{2\beta_B(t-\alpha)[9(1-\alpha)\beta_B-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(<0)$	$\frac{\partial p^{B*}}{\partial \beta_B} = -\frac{2\beta_A(t-\alpha)[9(1-\alpha)\beta_A-4]}{\{\beta_B(9(t-\alpha)\beta_A-2)-2\beta_A\}^2}(<0)$
$\frac{\partial p^{A*}}{\partial \beta_B} = \frac{2\beta_A(t-\alpha)[9(1-\alpha)\beta_A-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(>0)$	$\frac{\partial p^{B*}}{\partial \beta_A} = \frac{2\beta_B(t-\alpha)[9(1-\alpha)\beta_B-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(>0)$
$\frac{\partial D_j^{A*}}{\partial \beta_A} = -\frac{\beta_B[9(1-\alpha)\beta_B-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(<0)$	$\frac{\partial D_j^{B*}}{\partial \beta_B} = -\frac{\beta_A[9(1-\alpha)\beta_A-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(<0)$
$\frac{\partial D_j^{A*}}{\partial \beta_B} = \frac{\beta_A[9(1-\alpha)\beta_A-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(>0)$	$\frac{\partial D_j^{B*}}{\partial \beta_A} = \frac{\beta_B[9(1-\alpha)\beta_B-4]}{\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^2}(>0)$
$\frac{\partial \pi^{A*}}{\partial \beta_A} = -\frac{2[9(1-\alpha)\beta_B-4]^2\{2\beta_A+\beta_B[9(t-\alpha)\beta_A-2]\}}{9\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^3}$	$\frac{\partial \pi^{B*}}{\partial \beta_B} = -\frac{2[9(1-\alpha)\beta_A-4]^2\{2\beta_A+\beta_B[9(t-\alpha)\beta_A-2]\}}{9\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^3}$
$\frac{\partial \pi^{A*}}{\partial \beta_B} = \frac{4\beta_A[9(1-\alpha)\beta_B-4][9(1-\alpha)\beta_A-4][9(1-\alpha)\beta_A-2]}{9\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^3}$	$\frac{\partial \pi^{B*}}{\partial \beta_A} = \frac{4\beta_B[9(1-\alpha)\beta_B-4][9(1-\alpha)\beta_A-4][9(1-\alpha)\beta_B-2]}{9\{\beta_B[9(t-\alpha)\beta_A-2]-2\beta_A\}^3}$

4.8.6 Group discrimination

Proof of Proposition 18

Indifferent consumer and demand functions

The four consumer utilities corresponding to platforms $i \in \{A, B\}$ and sides $j \in \{1, 2\}$ are:

$$\begin{cases} u_1^A(x) = V_1^A + \alpha D_2^A - p_1^A - tx; \\ u_1^B(x) = V_1^B + \alpha D_2^B - p_1^B - t(1-x); \\ u_2^A(x) = V_2^A + \alpha D_1^A - p_2^A - tx; \\ u_2^B(x) = V_2^B + \alpha D_1^B - p_2^B - t(1-x). \end{cases}$$

The indifferent consumer on both sides of the market follows by setting:

$$u_1^A(\tilde{x}_1) = u_1^B(\tilde{x}_1) \Leftrightarrow V_1^A + \alpha D_2^A - p_1^A - t\tilde{x}_1 = V_1^B + \alpha D_2^B - p_1^B - t(1-\tilde{x}_1)$$
$$\Leftrightarrow \tilde{x}_1 = \frac{1}{2} + \frac{V_1^A - V_1^B + \alpha\left(2D_2^A - 1\right) - p_1^A + p_1^B}{2t},$$

and also:

$$u_2^A(\tilde{x}_2) = u_2^B(\tilde{x}_2) \Leftrightarrow V_2^A + \alpha D_1^A - p_2^A - t\tilde{x}_2 = V_2^B + \alpha D_1^B - p_2^B - t(1-\tilde{x}_2)$$
$$\Leftrightarrow \tilde{x}_2 = \frac{1}{2} + \frac{V_2^A - V_2^B + \alpha\left(2D_1^A - 1\right) - p_2^A + p_2^B}{2t}.$$

The demands as a function of prices and platforms' qualities are given by:

$$\begin{aligned}
D_1^A &= \frac{1}{2} + \frac{V_1^A - V_1^B + \alpha\left(2D_2^A - 1\right) - p_1^A + p_1^B}{2t} \\
&= \frac{1}{2} + \frac{V_1^A - V_1^B + \alpha\left\{2\left[\frac{1}{2} + \frac{V_2^A - V_2^B + \alpha\left(2D_1^A - 1\right) - p_2^A + p_2^B}{2t}\right] - 1\right\} - p_1^A + p_1^B}{2t} \\
\Leftrightarrow D_1^A &= \frac{1}{2} + \frac{\alpha\left(p_2^B - p_2^A\right) + t\left(p_1^B - p_1^A\right)}{2\left(t-\alpha\right)\left(t+\alpha\right)} + \frac{\alpha\left(V_2^A - V_2^B\right) + t\left(V_1^A - V_1^B\right)}{2\left(t-\alpha\right)\left(t+\alpha\right)}.
\end{aligned}$$

By symmetry, we obtain:

$$D_2^A = \frac{1}{2} + \frac{\alpha\left(p_1^B - p_1^A\right) + t\left(p_2^B - p_2^A\right)}{2\left(t-\alpha\right)\left(t+\alpha\right)} + \frac{\alpha\left(V_1^A - V_1^B\right) + t\left(V_2^A - V_2^B\right)}{2\left(t-\alpha\right)\left(t+\alpha\right)}.$$

Since total demand equals 1:

$$D_j^B = 1 - D_j^A.$$

Reaction functions

The profit functions are given by:

$$\begin{cases}
\pi^A(P) = p_1^A D_1^A(P) + p_2^A D_2^A(P); \\
\pi^B(P) = p_1^B D_1^B(P) + p_2^B D_2^B(P),
\end{cases}$$

Each platform i chooses p_1^i and p_2^i. The first-order conditions are given by:

$$\begin{cases}
\frac{\partial \pi^A}{\partial p_1^A} = 0 \\
\frac{\partial \pi^A}{\partial p_2^A} = 0 \\
\frac{\partial \pi^B}{\partial p_1^B} = 0 \\
\frac{\partial \pi^B}{\partial p_2^B} = 0
\end{cases}
\Leftrightarrow
\begin{cases}
-2tp_1^A - 2\alpha p_2^A + tp_1^B + \alpha p_2^B = -\left(t^2 - \alpha^2\right) - \alpha\left(V_2^A - V_2^B\right) - t\left(V_1^A - V_1^B\right); \\
-2\alpha p_1^A - 2tp_2^A + \alpha p_1^B + tp_2^B = -\left(t^2 - \alpha^2\right) - \alpha\left(V_1^A - V_1^B\right) - t\left(V_2^A - V_2^B\right); \\
tp_1^A + \alpha p_2^A - 2tp_1^B - 2\alpha p_2^B = -\left(t^2 - \alpha^2\right) + \alpha\left(V_2^A - V_2^B\right) + t\left(V_1^A - V_1^B\right); \\
\alpha p_1^A + tp_2^A - 2\alpha p_1^B - 2tp_2^B = -\left(t^2 - \alpha^2\right) + \alpha\left(V_1^A - V_1^B\right) + t\left(V_2^A - V_2^B\right).
\end{cases}$$

The solution of this system of equations yields the equilibrium prices (4.18), market shares (4.19) and profits (4.20).

Second-order conditions (Assumption 4.1)

The second-order conditions require:

$$|H_1| = \frac{\partial^2 \pi^A}{\partial {p_1^A}^2} = \frac{-2t}{2(t^2-\alpha^2)} = -\frac{t}{(t-\alpha)(t+\alpha)} < 0,$$

which is true.

Computing the second-order cross partial derivative, we obtain:

$$\frac{\partial^2 \pi^A}{\partial p_1^A \partial p_2^A} = \frac{-2\alpha}{2(t^2-\alpha^2)} = -\frac{2\alpha}{2(t-\alpha)(t+\alpha)}.$$

Note also that:

$$\frac{\partial^2 \pi^A}{\partial {p_2^A}^2} = -\frac{t}{(t-\alpha)(t+\alpha)}.$$

Finally, global profit maximization requires that:

$$|H_2| > 0 \Leftrightarrow \frac{t^2}{[(t-\alpha)(t+\alpha)]^2} > \frac{\alpha^2}{[(t-\alpha)(t+\alpha)]^2} \Leftrightarrow t > \alpha,$$

which is also true.

Positive prices and market shares (Assumption 4.3)

We, now, check $x_j^* \in [0,1]$. Substituting (4.18) and (4.19) into (4.17) we obtain:

$$x_j^* = \frac{1}{2} - \frac{\alpha\left(V_k^B - V_k^A\right) + t\left(V_j^B - V_j^A\right)}{6(t-\alpha)(t+\alpha)}.$$

(i) For $x_j^* > 0$, it is required that:

$$\frac{1}{2} - \frac{\alpha\left(V_k^B - V_k^A\right) + t\left(V_j^B - V_j^A\right)}{6(t-\alpha)(t+\alpha)} > 0 \;\Leftrightarrow\; \frac{\alpha\left(V_k^B - V_k^A\right) + t\left(V_j^B - V_j^A\right)}{(t-\alpha)(t+\alpha)} < 3.$$

With $q_k \equiv V_k^B - V_k^A$ and $q_j \equiv V_j^B - V_j^A$, the above condition is:

$$\alpha q_k + t q_j < 3(t-\alpha)(t+\alpha).$$

Considering both sides of the market:

$$\alpha q_2 + t q_1 < 3(t-\alpha)(t+\alpha); \tag{4.51}$$

$$\alpha q_1 + t q_2 < 3(t-\alpha)(t+\alpha). \tag{4.52}$$

Since $q_1 > q_2$ and $t > \alpha$:

$$\alpha q_2 + t q_1 > \alpha q_1 + t q_2, \tag{4.53}$$

which implies that the binding condition is (4.51).

(ii) Similarly, for $x_j^* < 1$:

$$\frac{1}{2} - \frac{\alpha\left(V_k^B - V_k^A\right) + t\left(V_j^B - V_j^A\right)}{6(t-\alpha)(t+\alpha)} < 1 \;\Leftrightarrow\; \frac{\alpha\left(V_k^B - V_k^A\right) + t\left(V_j^B - V_j^A\right)}{(t-\alpha)(t+\alpha)} > -3.$$

Since $q_k \equiv \left(V_k^B - V_k^A\right)$ and $q_j \equiv \left(V_j^B - V_j^A\right)$, the condition is:

$$\alpha q_k + t q_j > -3(t-\alpha)(t+\alpha).$$

It is satisfied because $q_1 > q_2 > 0$. $\square$

<u>Impacts of q_j on equilibrium profits</u>

$$\frac{\partial \pi^{A*}}{\partial q_1} = -\frac{1}{3} + \frac{\alpha q_2 + t q_1}{9(t-\alpha)(t+\alpha)}; \quad \frac{\partial \pi^{A*}}{\partial q_2} = -\frac{1}{3} + \frac{\alpha q_1 + t q_2}{9(t-\alpha)(t+\alpha)};$$

$$\frac{\partial \pi^{B*}}{\partial q_1} = \frac{1}{3} + \frac{\alpha q_2 + t q_1}{9(t-\alpha)(t+\alpha)}; \quad \frac{\partial \pi^{B*}}{\partial q_2} = \frac{1}{3} + \frac{\alpha q_1 + t q_2}{9(t-\alpha)(t+\alpha)}.$$

Then it follows:

(i) $\frac{\partial \pi^{A*}}{\partial q_1} < 0$ if and only if:

$$\alpha q_2 + t q_1 < 3(t-\alpha)(t+\alpha).$$

(ii) $\frac{\partial \pi^{A*}}{\partial q_2} < 0$ if and only if:

$$\alpha q_1 + t q_2 < 3(t-\alpha)(t+\alpha).$$

(iii) $\frac{\partial \pi^{B*}}{\partial q_1} > 0$ if and only if

$$\alpha q_2 + t q_1 > -3(t-\alpha)(t+\alpha).$$

(iv) $\frac{\partial \pi^{B*}}{\partial q_2} > 0$ if and only if

$$\alpha q_1 + t q_2 > -3(t-\alpha)(t+\alpha).$$

Given Assumption 4.2 and since $q_1 > q_2 > 0$, all the conditions are satisfied. $\square$

Proof of Proposition 19

Suppose that $q_1 = q + \varepsilon$ and $q_2 = q - \varepsilon$, with $q_1 \geq 0$, $q_2 \geq 0$ and $\varepsilon \in [0, q]$. Replacing in expressions (4.18), (4.19) and (4.20), we obtain the following equilibrium prices:

$$\begin{cases} p_1^{A*} = t - \alpha - \frac{q+\varepsilon}{3}; \quad p_2^{A*} = t - \alpha - \frac{q-\varepsilon}{3}; \\ p_1^{B*} = t - \alpha + \frac{q+\varepsilon}{3}; \quad p_2^{B*} = t - \alpha + \frac{q-\varepsilon}{3}. \end{cases} \tag{4.54}$$

The equilibrium market shares are given by:

$$\begin{cases} D_1^{A*} = \frac{1}{2} - \frac{\alpha(q-\varepsilon)+t(q+\varepsilon)}{6(t-\alpha)(t+\alpha)}; & D_2^{A*} = \frac{1}{2} - \frac{\alpha(q+\varepsilon)+t(q-\varepsilon)}{6(t-\alpha)(t+\alpha)}; \\ D_1^{B*} = \frac{1}{2} + \frac{\alpha(q-\varepsilon)+t(q+\varepsilon)}{6(t-\alpha)(t+\alpha)}; & D_2^{B*} = \frac{1}{2} + \frac{\alpha(q+\varepsilon)+t(q-\varepsilon)}{6(t-\alpha)(t+\alpha)}, \end{cases} \tag{4.55}$$

and the platforms' equilibrium profit are:

$$\begin{aligned} \pi^{A*} &= \frac{18t^3-18t^2\alpha-6t^2(q+\varepsilon)-6t^2(q-\varepsilon)-18t\alpha^2+t(q+\varepsilon)^2+t(q-\varepsilon)^2+18\alpha^3+6\alpha^2(q+\varepsilon)+6\alpha^2(q-\varepsilon)+2\alpha(q+\varepsilon)(q-\varepsilon)}{18(t^2-\alpha^2)}; \\ \pi^{B*} &= \frac{18t^3-18t^2\alpha+6t^2(q+\varepsilon)+6t^2(q-\varepsilon)-18t\alpha^2+t(q+\varepsilon)^2+t(q-\varepsilon)^2+18\alpha^3-6\alpha^2(q+\varepsilon)-6\alpha^2(q-\varepsilon)+2\alpha(q+\varepsilon)(q-\varepsilon)}{18(t^2-\alpha^2)}. \end{aligned} \tag{4.56}$$

Differentiating expressions (4.54), (4.55) and (4.56) with respect to ε gives us the respective impacts of an increase of the asymmetry of the perceived quality between the two sides. It follows that:

$$\begin{cases} \frac{\partial p_1^{A*}}{\partial \varepsilon} = -\frac{1}{3}; & \frac{\partial p_2^{A*}}{\partial \varepsilon} = \frac{1}{3}; \\ \frac{\partial p_1^{B*}}{\partial \varepsilon} = \frac{1}{3}; & \frac{\partial p_2^{B*}}{\partial \varepsilon} = -\frac{1}{3}. \end{cases} \tag{4.57}$$

The impacts on equilibrium market shares are given by:

$$\begin{cases} \frac{\partial D_1^{A*}}{\partial \varepsilon} = -\frac{1}{6(t+\alpha)}; & \frac{\partial D_2^{A*}}{\partial \varepsilon} = \frac{1}{6(t+\alpha)}; \\ \frac{\partial D_1^{B*}}{\partial \varepsilon} = \frac{1}{6(t+\alpha)}; & \frac{\partial D_2^{B*}}{\partial \varepsilon} = -\frac{1}{6(t+\alpha)}, \end{cases} \tag{4.58}$$

and on the platforms' equilibrium profits are given by:

$$\frac{\partial \pi^{A*}}{\partial \varepsilon} = \frac{\partial \pi^{B*}}{\partial \varepsilon} = \frac{2\varepsilon}{9\,(t+\alpha)}. \tag{4.59}$$

The payoff functions are convex relatively to $q_1 - q_2$, and this is the mathematical explanation behind the fact that a higher asymmetry of the quality gap across the sides of the market is profit enhancing for the active platforms. □

5.0 PRICE COMPETITION BETWEEN VERTI-ZONTALLY DIFFERENTIATED PLATFORMS

Abstract. We study a two-sided market duopoly in which the differentiated platforms compete in prices. Platforms are differentiated both vertically and horizontally (verti-zontal differentiation). We develop a model that is a synthesis of the two-sided market model of Armstrong (2006) [9] with Neven and Thisse (1990) [87], who study price competition between vertically and horizontally differentiated firms. Considering a symmetric baseline model, in which price discrimination between sides cannot take place, we find that equilibrium outcomes depend on the strength of the inter-group network effects vis-à-vis the magnitude of the intrinsic quality differences between the platforms. Moreover we find that, under horizontal dominance, the profit of the low-quality platform may decrease (or increase) as the quality of the better product improves. This result contrasts with the seminal contribution of Shaked and Sutton (1982) [105]. We also conclude that, whatever the type of dominance, the intensity of the network effect has the same impact on equilibrium prices and profits and the profit of both platforms is decreasing with the intensity of the network effect, which confirms Armstrong (2006) [9].

Keywords: Two-sided markets, Horizontal differentiation, Vertical differentiation.

JEL Classification Numbers: D42, D43, L13.

5.1 INTRODUCTION

In two-sided markets, firms (platforms) are intermediaries between two categories of agents, such that at least one category of agents generates inter-group network effects over the other group of agents in the market. Accordingly, in these markets, the value of participating in a certain platform (for a certain category of agents) depends not only on its intrinsic characteristics but also on the number of agents (of a different category) participating in the same platform.

Examples of two-sided platforms include: (*i*) media outlets, which allow for the interaction between advertisers and readers; (*ii*) clubs, which allow for the interaction between men and women; (*iii*) e-commerce platforms, which allow for the interaction between buyers and sellers, and so on.

In the context of the examples above, real life reveals situations in which platforms are simultaneously vertically and horizontally differentiated [1]. In the case of clubs, we observe that they are horizontally differentiated (including the type of music they play or their environment) as well as vertically differentiated (quality of the air, quality of the food, parking space). Similarly, in the case of e-commerce platforms, they often differ along horizontal dimensions (such as their aesthetics, the type of products they sell) as well as vertical dimensions (like their usability, delivery times and advertising intensity).

In this paper, we aim at investigating the nature of price competition between platforms that are both horizontally and vertically differentiated. Using the terminology proposed by Di Comite, Thisse and Vandenbussche (2011) [35], in the context of monopolistic competition, we may say that our objective is to analyze price competition between verti-zontally differentiated platforms.

Following the seminal works of Rochet and Tirole (2003) [94] and Armstrong (2006) [9], a vast literature has analyzed the strategic aspects of competition between two-sided platforms. In particular, following Armstrong (2006) [9], a considerable number of works has been devoted to the analysis of strategic interaction between horizontally differentiated

[1] See Gabszewicz and Resende (2012) [56] for further details concerning horizontal and vertical differentiation in the press industry.

platforms (see, for example, Zacharias and Serfes (2012) [104] and Amir, Gabszewicz and Resende (2013) [4]).

In this paper, we combine two strands of literature: the literature on price competition under horizontal and vertical differentiation and the literature on two-sided markets.

Concerning the first line of literature, Economides (1989) [39] and Neven and Thisse (1990) [87] both analyze a two-dimensional vertical and horizontal differentiation model in which firms compete on quality, variety and price. Economides (1989) [39] assumes that the variety (horizontal) choice takes place before the quality (vertical) choice. Considering that marginal costs are increasing in quality levels, he concludes that in equilibrium, we observe maximum variety differentiation and minimum quality differentiation. Neven and Thisse (1990) [87] consider a three-stage game, in which firms first choose their quality, afterwards their location, competing in prices at the end of the game. Normalizing marginal costs to zero, the authors find an equilibrium that exhibits maximum differentiation on one dimension and minimum differentiation on the other. The maximally differentiated dimension can either be the quality or variety dimension. More recently, other authors have addressed the issue of price competition in conventional markets with horizontal or/and vertical differentiation (see, for example, Levin, Peck and Ye (2009) [82], Gabsewicz and Resende (2012) [56], Gabszewicz and Wauthy (2014) [58]). The present paper contributes to this literature by investigating how the nature of price competition between verti-zontally differentiated firms can be affected by inter-group network effects.

Our paper also contributes to the literature on two-sided markets. Most of the literature on this field has focused on the waterbed effects and tipping prevention (see for example, Rochet and Tirole (2003) [94] and Armstrong (2006) [9]). A considerable literature has also addressed the issue of price competition between horizontally differentiated platforms (see for example, Zacharias and Serfes (2012) [104] and Amir, Gabszewicz and Resende (2013) [4]). Using the Hotelling framework proposed by Armstrong (2006) [9], these works have studied pricing competition between horizontally differentiated platforms. When agents are allowed to participate in only one platform, these papers (including Armstrong (2006) [9]) show that the existence of inter-group network effects often intensifies price competition between platforms.

To the best of our knowledge, Amir, Gabszewicz and Resende (2013) [4] is the first paper that simultaneously combines horizontal and vertical differentiation in two-sided markets. However, the authors do not analyze price competition between the two-sided platforms. We contribute to this literature by analyzing how the impact of inter-group network effects on equilibrium outcomes depends on the interplay of vertical and horizontal differentiation.

In our model, we bring together the verti-zontal differentiation set-up developed by Neven and Thisse (1990) [87] and the two-sided market framework proposed by Armstrong (2006) [9]. Our motivation is to understand if the standard results of two-sided markets are disrupt with the introduction of the concepts of horizontal dominance and vertical dominance that emerge from Neven and Thisse (1990) [87].

Platforms are assumed to be both horizontally and vertically differentiated. On the horizontal dimension, we consider a Hotelling framework in line with the two previous papers. Following Armstrong (2006) [9], we consider that firms are exogenously located at the extremes of the Hotelling line. As in Neven and Thisse (1990) [87] platforms are also vertically differentiated and consumers have heterogeneous willingness to pay for quality. In addition, we allow for inter-group network effects, with the objective of studying how they affect pricing strategies of verti-zontally differentiated firms.

In the baseline model, we do not allow for price discrimination between sides (as an illustrative example, the reader may consider of a club where men and women pay the same entry fee, or a e-commerce platform, where sellers and buyers are required to pay the same access fee in order to participate in the platform). The paper proposes a two-stage game with the following structure. In the first stage, platforms set their access prices and afterwards consumers decide which platform they are willing to join.

Our equilibrium analysis shows that the high-quality platform sets a higher access price, serves a larger market share and earns a higher profit than the low-quality platform, in line with standard vertical differentiation literature (see, for example, Tirole (2003) [109]). We show that equilibrium outcomes depend on the strength of the inter-group network effects *vis-à-vis* the magnitude of the intrinsic quality differences between the platforms. Traditional literature that combines simultaneously horizontal and vertical differentiation argues that revenues of both firms increase as the quality of the better product improves (see Shaked and Sutton, 1982 [105]).

Our finding establishes that this argument may be disrupted under horizontal dominance. We find that, under pure horizontal dominance, the profit of the low-quality platform decreases as the quality of the better product improves. This result contrasts with the seminal contribution on vertical differentiation of Shaked and Sutton (1982) [105]. In addiction, whatever the type of dominance, we find that the intensity of the network effect has the same impact on equilibrium prices and profits and the profit of both platforms is decreasing with the intensity of the network effect, which confirms Armstrong (2006) [9].

The rest of the paper is organized as follows. In the next section 5.2 we present the model. Section 5.3 analyzes the model and section 5.4 characterizes the equilibrium outcomes. Section 5.5 performs a comparative statics. Finally section 5.6 concludes. The proofs of the Propositions and Lemmas are relegated to the Appendix.

5.2 THE MODEL

Consider a two-sided market with two platforms, $i \in \{A, B\}$, and two sides, $j \in \{1, 2\}$. The platforms operate with zero marginal costs[2] and they are both horizontally and vertically differentiated. As in Neven and Thisse (1990) [87], we assume that the platforms differ in two characteristics: (i) their location on the Hotelling line (horizontal differentiation), and (ii) their intrinsic quality (vertical differentiation).

In particular, platform A is located at the left extreme of the Hotelling line $\left(x^A = 0\right)$, whereas platform B is located at the right extreme $\left(x^B = 1\right)$. The intrinsic quality of platform A is denoted by q^A, whereas the intrinsic quality of platform B is q^B, with $q^B > q^A$. Then, platform A is the low-quality platform while platform B is the high-quality platform.

In each side of the market, there is a unit mass of heterogeneous consumers. They differ on their location, x, as well as on their quality valuation, y. Accordingly, in side j, each consumer is identified by the pair (x, y). As in Neven in Thisse (1990) [87], we assume

[2] Our results would remain unchanged under constant and symmetric marginal costs. A similar assumption has been adopted by Neven and Thisse (1990) [87], among others.

that in each side of the market consumers are uniformly distributed over the unit square $[0, 1] \times [0, 1]$ and we normalize their transportation costs to 1.

We depart from Neven and Thisse (1990) [87] by introducing the possibility of positive inter-group externalities. As in Serfes and Zacarias (2012) [104], we assume that the intensity of inter-group externalities is the same on both sides of the market, being measured by the parameter $\alpha \geq 0$.

In the market side $j \in \{1, 2\}$, a consumer type $(x, y) \in [0, 1] \times [0, 1]$ obtains the following utility from participating in platform A:

$$u_j^A(x, y) = v + yq^A + \alpha D_k^A - p^A - x,$$

where $v \in \mathbb{R}_+$ is sufficiently high for the market to be covered in equilibrium, D_k^A is the demand of platform A on the other side ($k \neq j$) and p^A is the access price charged by platform A, which applies to both sides of the market (price discrimination between sides is not allowed).[3]

Analogously, when participating in platform B, a side-j consumer of type (x, y) obtains the following utility:

$$u_j^B(x, y) = v + yq^B + \alpha D_k^B - p^B - (1 - x)\,,$$

where D_k^B is the demand of platform B on the other side ($k \neq j$) and p^B is the access price charged by platform B to both sides of the market.

In order to study equilibrium outcomes when platforms are simultaneously horizontally and vertically differentiated, we consider a game where platforms simultaneously set access prices for both sides and consumers in each side simultaneously decide which platform to join.

Throughout the paper, we assume that the inter-group externality is relatively weak in order to avoid market tipping.

Assumption 5.1 (Weak inter-group externality) *The inter-group externality is relatively weak:* $\alpha < 1$.

[3] If price discrimination was allowed, we would expect the symmetry of the model to induce no discrimination in equilibrium.

5.3 DEMAND AND PROFITS

In side j, the consumer type $(\widetilde{x}_j(y), y)$ for whom

$$u_j^A(\widetilde{x}_j(y), y) = u_j^B(\widetilde{x}_j(y), y)$$

is indifferent between participating in platform A or platform B. Solving the previous equation for $\widetilde{x}_j(y)$, we obtain that, for a given y, the indifferent consumer in side j is located at:

$$\widetilde{x}_j(y) = \tfrac{1}{2} + \tfrac{p^B - p^A}{2} + \tfrac{\alpha(D_k^A - D_k^B)}{2} - \tfrac{q}{2}y, \tag{5.1}$$

where q represents the quality gap, i.e. $q \equiv q^B - q^A$.

Taking into consideration that the market is fully covered, $D_k^B = 1 - D_k^A$ and that consumers formulate rational expectations about the demand on the other side of the market, with $D_k^A = \int_0^1 \widetilde{x}_k(y)\,dy$ for $\widetilde{x}(y_j) \in [0,1]$,[4] equation (5.1) can be re-written as follows:

$$\widetilde{x}_j(y) = \frac{1}{2} + \frac{p^B - p^A}{2} + \frac{\alpha(2D_k^A - 1)}{2} - \frac{q}{2}y, \tag{5.2}$$

or equivalently:

$$\widetilde{y}_j(x) = \frac{1}{q} + \frac{p^B - p^A}{q} + \frac{\alpha(2D_k^A - 1)}{q} - \frac{2}{q}x. \tag{5.3}$$

The previous equations show that for a given vector of prices $\left(p^A, p^B\right)$ the position of the indifferent consumer $\widetilde{y}_j$ evolves linearly and negatively with x, as $\frac{\partial \widetilde{y}_j(x)}{\partial x} = -\frac{2}{q} < 0$.

Consider two types of consumers: (x, y) and (x', y'), with $x' < x$, meaning that the second consumer is located closer to firm A. The two consumers can only be both indifferent between participating in platform A and B if the second consumer has a higher willingness to pay for quality than the first one i.e. $y' > y$.

Note that, if, for a given x, we have $\widetilde{y}_j(x) \in [0,1]$, for such x, side j consumers located at $y_j \in [0, \widetilde{y}_j(x)]$ participate in platform A, whereas those located at $y_j \in (\widetilde{y}_j(x), 1]$ participate in platform B. In other words, consumers above the line (5.3) participate in platform B, whereas those below that line prefer to participate in platform A. If we have $\widetilde{y}(x_j) < 0$, $\forall$

[4] This formulation means that an agent of a side j $(j = 1, 2)$ can interact with an agent of an opposite side k ($k = 1$ when $j = 2$ or $k = 2$ when $j = 1$), for any given variety x and quality y within the square $[0,1] \times [0,1]$.

$x \in [0,1]$, resp. $\widetilde{y}(x_j) > 1$, $\forall\, x \in [0,1]$, then, all consumers in side j buy from platform B, resp. platform A.

The figure 7 below illustrates the choice of consumers between the two platforms for different pairs of prices $\left(p^A, p^B\right)$.

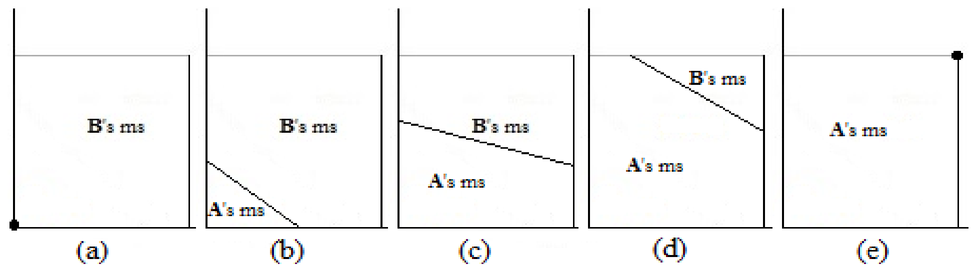

Figure 7: Consumers' choice between platforms under vertical dominance [variety x - horizontal axis; quality y - vertical axis; ms - standing for market share]

Considering the position of the indifferent consumer line, it is now possible to obtain the analytical expressions of the demands of platform A and platform B, in side j, as a function of firms´ access prices, p^A and p^B.

As in Neven in Thisse (1990) [87], in order to obtain these analytical expressions we first need to distinguish two cases according to the relative importance that consumers attach to vertical differentiation *vis-à-vis* horizontal differentiation.

Definition 20. ***(Horizontal Dominance versus Vertical Dominance)***

(i) Horizontal dominance corresponds to an environment where the intrinsic quality gap between the platforms, q, is relatively low, in particular, $q < 2$.

(ii) Instead, when the intrinsic quality gap is sufficiently large, $q > 2$, vertical dominance prevails.

Proof. As in Neven and Thisse (1990) [87], horizontal (resp. vertical) dominance arises when $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| > (<)1$, or equivalently, $q < (>)2$. □

In order to better understand the intuition and the need for the distinction pointed out in Definition 20, consider the two cases of figure 8.

For the values of the parameters considered in figure $8(b)$, we have $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| < 1$, or equivalently $q > 2$. As in this case, the intrinsic quality gap between the two firms is relatively large, Neven and Thisse (1990) [87] introduce the concept of vertical dominance to denote the domain of parameters for which $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| < 1$.

However, when $q < 2$, or equivalently, $\left|\frac{\partial \widetilde{y}(x)}{\partial x}\right| > 1$, the intrinsic quality gap between the firms is relatively low and horizontal dominance prevails. In that case, for the same price vector $\left(p^A, p^B\right)$ we would obtain a different demand configuration as illustrated in the figure $8(a)$.

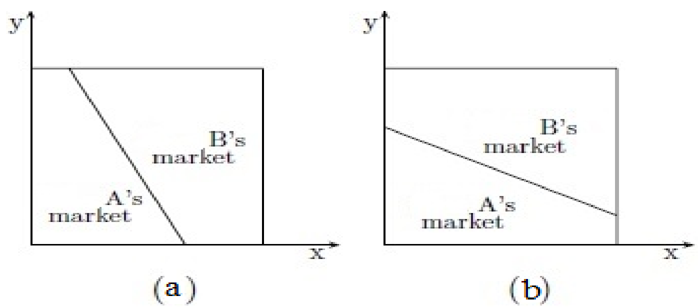

Figure 8: Horizontal (a) versus vertical dominance (b)

In light of this, for each side of the market j, we now define the analytical expressions of the platforms' demands as a function of their access prices $\left(p^A, p^B\right)$, under horizontal and vertical dominance[5].

We denote by $D_j^{i|VD}\left(p^A, p^B\right)$ the demand of platform i in side j, under vertical dominance, with:

$$D_j^{A|VD}\left(p^A, p^B\right) =$$

$$\begin{cases} 0 & if \quad p^A - p^B > 1-\alpha; \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{2\alpha^2} & if \quad -(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha; \\ \frac{1}{q-2\alpha}\left(p^B - p^A - \alpha\right) & if \quad 1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q}; \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}}{2\alpha^2} & if \quad -q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\frac{2\alpha}{q}; \\ 1 & if \quad p^A - p^B < -q-(1-\alpha), \end{cases} \tag{5.4}$$

[5]See Appendix A for further details on the computation of the analytical expression of the demand functions of each platform, under horizontal and vertical dominance.

and $D_j^{B|VD}\left(p^A,p^B\right)=1-D_j^{A|VD}\left(p^A,p^B\right)$. Analogously, we denote by $D_j^{i|HD}\left(p^A,p^B\right)$, the demand of platform i in side j, under horizontal dominance, with:

$$D_j^{A|HD}\left(p^A,p^B\right)=$$

$$\begin{cases} 0 & if \;\; p^A-p^B>1-\alpha \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{2\alpha^2} & if \;\; 1-\alpha-q+\frac{q\alpha}{2}<p^A-p^B\leq 1-\alpha \\ \frac{1}{2}\left(1+\frac{p^B-p^A}{1-\alpha}-\frac{q}{2(1-\alpha)}\right) & if \;\; -(1-\alpha)-\frac{q\alpha}{2}\leq p^A-p^B\leq 1-\alpha-q+\frac{q\alpha}{2} \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\,]}}{2\alpha^2} & if \;\; -q-(1-\alpha)\leq p^A-p^B<-(1-\alpha)-\frac{q\alpha}{2} \\ 1 & if \;\; p^A-p^B<-q-(1-\alpha) \end{cases}. \tag{5.5}$$

and $D_j^{B|HD}\left(p^A,p^B\right)=1-D_j^{A|HD}\left(p^A,p^B\right)$. The demand functions $D_j^{A|VD}\left(p^A,p^B\right)$ and $D_j^{A|HD}\left(p^A,p^B\right)$ must be decreasing and continuous functions of the platforms' own prices. However, they are not globally concave [6]. In the particular case of vertical dominance, the demand is linear for price vectors $\left(p^A,p^B\right)$ such that:

$$1+\alpha-q-\frac{2\alpha}{q}\leq p^A-p^B\leq-(1+\alpha)+\frac{2\alpha}{q}.$$

When $-(1+\alpha)+\frac{2\alpha}{q}<p^A-p^B\leq 1-\alpha$, the demand $D_j^{A|VD}\left(p^A,p^B\right)$ is strictly convex in p^A. Analogously, when:

$$-q-(1-\alpha)\leq p^A-p^B<1+\alpha-q-\frac{2\alpha}{q},$$

the demand $D_j^{B|VD}\left(p^A,p^B\right)$ is strictly convex in p^B. In the particular case of horizontal dominance, the demand is linear for price vectors $\left(p^A,p^B\right)$ such that:

$$-(1-\alpha)-\frac{q\alpha}{2}\leq p^A-p^B\leq 1-\alpha-q+\frac{q\alpha}{2}.$$

When $1-\alpha-q+\frac{q\alpha}{2}\leq p^A-p^B<1-\alpha$, the demand $D_j^{A|HD}\left(p^A,p^B\right)$ is strictly convex in p^A. Analogously, when:

$$-q-(1-\alpha)\leq p^A-p^B<-(1-\alpha)-\frac{q\alpha}{2},$$

[6] The formal proof of these properties is straightforward, given the analytical expressions of demands (5.4) and (5.5).

the demand $D_j^{B|HD}\left(p^A, p^B\right)$ is strictly convex in p^B. Given the demands faced by each platform in each sided of the market, it is now possible to obtain the profit of the two platforms as follows:

$$\pi^i(p^A, p^B) = p^i D_1^i(p^A, p^B) + p^i D_2^i(p^A, p^B),$$

where $D_j^i(p^A, p^B) = D_j^{i|VD}\left(p^A, p^B\right)$, in the case of vertical dominance, and $D_j^i(p^A, p^B) = D_j^{i|HD}\left(p^A, p^B\right)$, in the case of horizontal dominance. Note that the equations (5.4) and (5.5) imply symmetric demands in the two sides of the market:

$$D_1^i(p^A, p^B) = D_2^i(p^A, p^B) = D^i(p^A, p^B),$$

thus, the platforms' profit can be computed as follows:

$$\pi^i(p^A, p^B) = 2p^i D^i(p^A, p^B),\ i \in \{A, B\}.$$

Since the demand functions $D^i(p^A, p^B)$ are not globally concave, it is necessary to study under which conditions the profit of each platform is quasi-concave with respect to its access price. Caplin and Nalebuff (1991) [22] for the case of product differentiation in multiple dimensions and, in particular, Neven and Thisse (1990) [87] for the case of product differentiation in two dimensions, provide sufficient conditions for the existence of a quasi-concave payoff function of the active firms in the market. Without network externalities, our manuscript reduces to a model which satisfies the assumptions in Neven and Thisse (1990) [87]. Since our model incorporates network effects, that are present for any given level of quality and variety, we analyse quasi-concavity of the profit functions.

Lemma 21. ***(Quasi-concavity of the profit function)***

For any pair combination (α, q), the profit functions of platforms A and B are quasi-concave in p^A and p^B both under vertical and horizontal dominance, respectively.

Proof. See Appendix B. □

Given Lemma 21, the first order conditions of the platforms' maximization problem intersect at least once which guarantees that the first order conditions associated with the profit maximization problem are sufficient to characterize firms' optimal pricing choices. We also examined the shape of the profit functions for a grid of values of the relevant parameters. Figures $9(a)$ and $9(b)$ illustrate a possible configuration of the demand function of low-quality platform A and figures $10(a)$ and $10(b)$ present a possible configuration of the profit function of the low-quality platform A, for different levels of the inter-group externality under horizontal and vertical dominance, respectively.

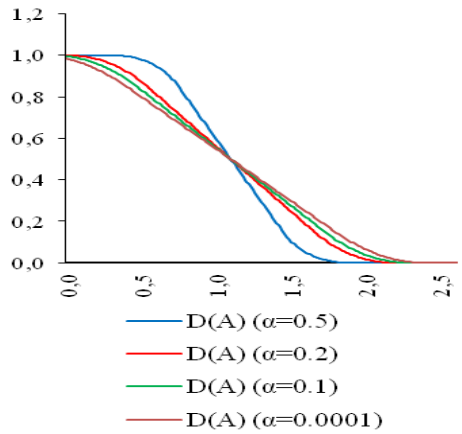

Figure 9(a) - D^A (y-axis) as a function of p^A (x-axis)

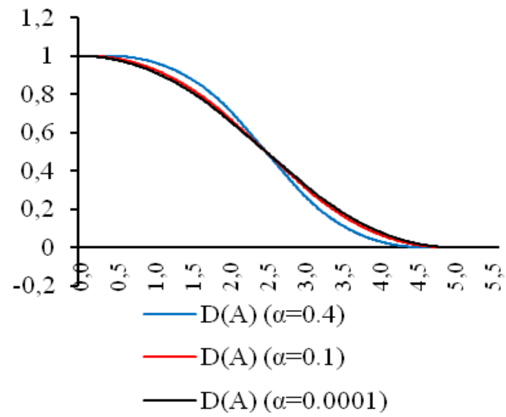

Figure 9(b) - D^B (y-axis) as a function of p^B (x-axis)

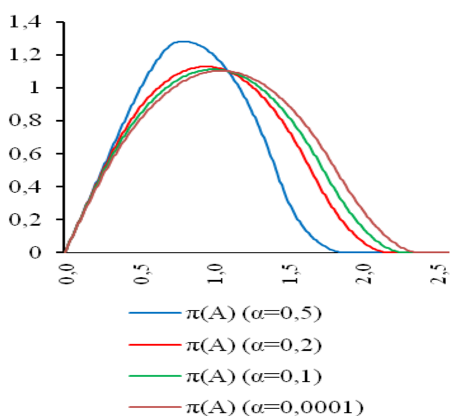

Figure 10(a) - π^A (y-axis) as a function of p^A (x-axis)

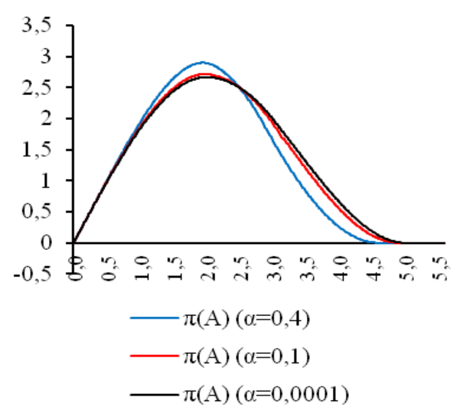

Figure 10(b) - π^B (y-axis) as a function of p^B (x-axis)

The above figures illustrate that the payoff functions are quasi-concave on its prices domain.

5.4 ANALYSIS

Given that platforms take decisions on access prices anticipating consumers behavior, the problem of platform i writes as follows:

$$\max_{p_i} 2p^i D^i(p^A, p^B),$$

with $i = A, B$ and $D^i(p^A, p^B) = D^{A|VD}\left(p^A, p^B\right)$ in the case of vertical dominance, and $D^i(p^A, p^B) = D^{A|HD}\left(p^A, p^B\right)$ in the case of horizontal dominance.

Then, in line with Neven and Thisse (1990) [87], we should determine the equilibrium prices as a function of the product characteristics. This leads us to distinguish between six types of equilibrium candidates, namely three under vertical dominance and three under horizontal dominance. By other words, for each kind of dominance, we have the following types of equilibrium:

(i) *case/subgame 1* - the equilibrium occurs on the linear segments of D^A and D^B;

(ii) *case/subgame 2* - the equilibrium occurs in the strictly convex segment of D^A and in the strictly concave segment of D^B;

(iii) *case/subgame 3* - the equilibrium occurs in the strictly concave segment of D^A and in the strictly convex segment of D^B.

We start by studying the price stage equilibrium for the linear segment of the demand of platforms A and B, verifying Assumption 5.1.

5.4.1 Pure horizontal dominance equilibrium

We first describe the region where the equilibrium candidate under horizontal dominance exists, corresponding to an equilibrium occurring under the circumstances described at figure 8(a) and, then, we fully characterize the correspondent equilibrium candidate.

Proposition 22. ***(Equilibrium under pure horizontal dominance)***

Let $q \leq \frac{6(1-\alpha)}{4-3\alpha}$. The equilibrium outcomes are described as follows:

$$\begin{cases} p^{A^*} = 1-\alpha-\frac{q}{6},\ p^{B^*} = 1-\alpha+\frac{q}{6}; \\ D^{A^*} = \frac{1}{2}\left[1-\frac{q}{6(1-\alpha)}\right],\ D^{B^*} = \frac{1}{2}\left[1+\frac{q}{6(1-\alpha)}\right]; \\ \pi^{A^*} = \frac{1}{36}\frac{[q-6(1-\alpha)]^2}{1-\alpha},\ \pi^{B^*} = \frac{1}{36}\frac{[q+6(1-\alpha)]^2}{1-\alpha}. \end{cases}$$

Proof. See Appendix B. □

Comparing p^{A^*} and p^{B^*}, it can be checked that for any (α, q) that verifies the condition of Proposition 22 we obtain $p^{B^*} > p^{A^*}$, meaning that the access fee quoted by the high-quality platform is always higher than the access fee charged by the low-quality platform.

In horizontal dominance, equilibrium prices are affected by the intensity of inter-group network effects.

It is also worth noting that the high-quality platform always has a higher market share than the low-quality platform, for any (α, q) that verifies the condition of Proposition 22. Since the former platform also charges a higher price than the latter platform, we always have $\pi^{B^*} > \pi^{A^*}$.

5.4.2 Pure vertical dominance equilibrium

We first describe the region where the equilibrium candidate under vertical dominance exists, corresponding to an equilibrium occurring under the circumstances described at figure 8(b). Then, we fully characterize the correspondent equilibrium candidate.

Proposition 23. ***(Equilibrium under pure vertical dominance)***

Let $q \geq \frac{3}{2}\alpha + \frac{1}{2}\sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3} + \frac{3}{2}$. *The equilibrium outcomes are described as follows:*

$$\begin{cases} p^{A^*} = \frac{q}{3} - \alpha, \ p^{B^*} = \frac{2q}{3} - \alpha; \\ D^{A^*} = \frac{q-3\alpha}{3(q-2\alpha)}, \ D^{B^*} = \frac{2q-3\alpha}{3(q-2\alpha)}; \\ \pi^{A^*} = \frac{2(q-3\alpha)^2}{9(q-2\alpha)}, \ \pi^{B^*} = \frac{2(2q-3\alpha)^2}{9(q-2\alpha)}. \end{cases}$$

Proof. See Appendix B. □

Comparing p^{A^*} and p^{B^*}, it can be easily checked that for any (α, q), we always obtain $p^{B^*} > p^{A^*}$, meaning that the access fee quoted by the high-quality platform is always higher than the access fee charged by the low-quality platform. In vertical dominance, equilibrium prices are affected by the intensity of inter-group network effects. It is also worth noting that the high-quality platform always has a higher market share than the low-quality platform, for any (α, q). Since the former also charges a higher price than the latter, we always have $\pi^{B^*} > \pi^{A^*}$.

5.4.3 Verti-zontal equilibrium

After characterizing the equilibrium candidates on the linear segments of the demands of platforms A and B (*case 1*), we verify the other interior equilibrium candidates from case 2, for each type of dominance, corresponding to the circumstances described in figure 7(b).[7]. Figure 11 shows the price domain conditions under which pure vertical dominance occurs (corresponding to the blue region, characterized in Proposition 23) and under which pure horizontal dominance takes place (corresponding to the yellow region, characterized in Proposition 22), where each equilibrium candidate is an effective equilibrium. The green region corresponds to the verti-zontal equilibrium candidate, yielding a market share configuration as the one described in figure 9(b).

[7]As Neven and Thisse (1990) [87] point out, since platform A is the low-quality platform, a situation described as in figure 9(d) never arises at the equilibrium. This implies, in our manuscript, that under horizontal dominance for $-(1-\alpha)-q \leq p^A - p^B < (1-\alpha)-q$ and under vertical dominance for $-q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\frac{2\alpha}{q}$, there is no price equilibrium since the prices solving the first order conditions are incompatible with the corresponding price domain condition.

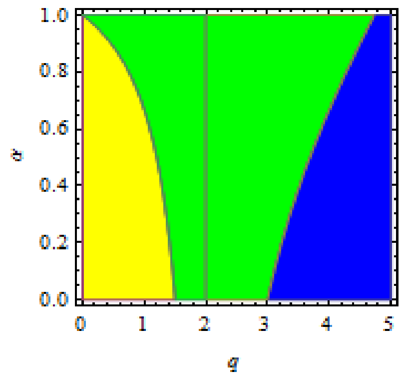

Figure 11 - Parameter regions corresponding to pure horizontal dominance (yellow), intermediate quality gap (green) and pure vertical dominance (blue)

For such values, the equilibrium occurs in the strictly convex segment of D^A and in the strictly concave segment of D^B. The profit of platform A and platform B are given by:

$$\pi^A = \frac{p^A\left[q - \alpha(1-\alpha+p^B-p^A) - \sqrt{q\left[q-2\alpha(1-\alpha+p^B-p^A)\right]}\right]}{\alpha^2},$$

$$\pi^B = 2p^B\left\{1 - \frac{\left[q - \alpha(1-\alpha+p^B-p^A) - \sqrt{q\left[q-2\alpha(1-\alpha+p^B-p^A)\right]}\right]}{2\alpha^2}\right\}.$$

Given the quasi-concavity of the profit function, an equilibrium price candidate is always an equilibrium as long as the price domain condition is verified. Under horizontal dominance, the equilibrium is verified for:

$$1-\alpha-q+\tfrac{q\alpha}{2} < p^{A*}-p^{B*} \leq 1-\alpha,$$

and under vertical dominance, the equilibrium is verified for:

$$-(1+\alpha)+\tfrac{2\alpha}{q} < p^{A*}-p^{B*} \leq 1-\alpha.$$

We are not able to analytically solve the model for this "verti-zontal" equilibrium candidate[8].

However, the theorem of Debreu, Glicksberg and Fan (1952) allows us to conclude that an equilibrium exists for this case (see, for example, pp. 34, of Fudenberg and Tirole, 1991 [52]).

[8]See Appendix C for details on the best reply functions of both platforms.

Theorem 24. ***(Debreu, Glicksberg and Fan (1952))*** *Consider a strategic-form game: (i) whose strategy spaces S_i are nonempty compact convex subsets of an Eucledian space. If the payoff functions are (ii) continuous and (iii) quasi-concave in s_i, there exists a pure-strategy Nash equilibrium.*[9]

Lemma *21* guarantees that the payoff functions are quasi-concave and, thus, the previous theorem can be applied.

5.5 COMPARATIVE STATICS

In this section we analyze how the magnitude of the quality gap q, and the intensity of the inter-group externality α, affect equilibrium outcomes.

5.5.1 Impact of the quality gap

Regarding the impact of the quality gap on the equilibrium prices, we obtain that the price of the high-quality platform always increases with the magnitude of the quality gap under horizontal and under vertical dominance. Everything else the same, an increase in q makes platform B relatively more attractive than platform A and, therefore, platform B is able to increase its price for all (q, α).

As far as concerns platform A, under horizontal dominance, an increase in q reduces its price.

Since q is not too high (meaning that products are not very differentiated on the vertical dimension), the low-quality firm reduces its price after an increase in q. In contrast, under pure vertical dominance, the price of the low-quality gap also increases with q. In that case, products are already significantly differentiated in the vertical dimension and, thus, an increase in q makes products even more differentiated, relaxing price competition, which is line with standard vertical differentiation literature (see, for example, Shaked and Sutton (1982) [105]).

[9] Debreu (1952) [31] used a generalization of this theorem to prove that competitive equilibria exist when consumers have quasi-convex preferences.

Despite the fact that both prices are increasing with q, the price of the high-quality platform always increases more than the price of its rival since $\frac{\partial p^{B^*}}{\partial q} > \frac{\partial p^{A^*}}{\partial q} > 0$.

Regarding the impact of the quality gap on equilibrium market shares, we obtain that under horizontal dominance $\frac{\partial D^{A^*}}{\partial q} = -\frac{1}{12(1-\alpha)} < 0$, implying $\frac{\partial D^{B^*}}{\partial q} > 0$ (since the market is fully covered). Following an increase in q, the high-quality platform becomes more attractive than the low-quality platform and it is able to increase its market share, despite the fact that the price gap $p^{B^*} - p^{A^*}$ is increasing with q.

However, when the quality gap is very large so that we are in the vertical dominance case, it follows $\frac{\partial D^{A^*}}{\partial q} = \frac{\alpha}{3(q-2\alpha)^2} > 0$, implying $\frac{\partial D^{B^*}}{\partial q} < 0$. This is so, because the high-quality platform increases its price substantially more than the low-quality platform.

Lemma 4 (pp.8) in Shaked and Sutton (1982) [105] establishes that "*revenues of both firms increase as the quality of the better product improves*".

Considering the pure horizontal dominance environment and evaluating the impact of q on the equilibrium profits of platforms A and B, follows that:

$$\frac{\partial \pi^{A^*}}{\partial q} < 0; \ \frac{\partial \pi^{B^*}}{\partial q} > 0,$$

Then, under horizontal dominance, the revenue of the high-quality platform B increases as the quality of the better product improves and the revenue of the low-quality platform A decreases as the quality of the better product improves. This result is not only distinct from Shaked and Sutton (1982) but also complements Neven and Thisse (1990) [87] since they did not take into consideration the presence of network effects influencing simultaneously both variety and quality.

Our results are similar to Shaked and Sutton (1982) [105] under vertical dominance. The equilibrium profit of the low-quality platform A is always increasing with q (since the equilibrium price and market shares are increasing with q). In the high-quality platform B, the market shares are decreasing in q. However, the positive price effect more than compensates its negative demand effect.

We also find that when the inter-group externality is sufficiently high and the quality gap is sufficiently low but within the vertical dominance domain (a low subdomain of vertical dominance), the positive price effect does not compensate the negative demand effect on the high-quality platform B.

Lemma 25. ***(Impact of vertical differentiation)***

Under pure horizontal dominance, follows that $\frac{\partial \pi^{A^*}}{\partial q} < 0$.

Proof. *See Appendix B.* □

5.5.2 Impact of the intensity of the inter-group externality

In what concerns the impact of the intensity of the inter-group network effect on equilibrium prices, we obtain that the standard results in two-sided markets hold. Whatever the type of dominance, we get that:

$$\frac{\partial p^{A^*}}{\partial \alpha} = \frac{\partial p^{B^*}}{\partial \alpha} = -1.$$

In line with Armstrong (2006) [9], when the inter-group network effect is sufficiently weak consumers benefit from lower prices due to inter-group network effects (when these effects take place, platforms set discounts in one side of the market with the objective of enhancing demand in opposite side of the market).

Regarding the impact of the inter-group network intensity on equilibrium market shares, we obtain that for horizontal dominance yields $\frac{\partial D^{A^*}}{\partial \alpha} < 0$ implying $\frac{\partial D^{B^*}}{\partial \alpha} > 0$, since $D^{B^*} = 1 - D^{A^*}$. More precisely, $\frac{\partial D^{A^*}}{\partial \alpha} = -\frac{1}{12}\frac{q}{(\alpha-1)^2} < 0$. Hence, an increase on the intensity of inter-group network externalities increases market concentration, favoring the high-quality platform. When α increases, consumers value more the number of consumers (on the other side) participating in each platform and therefore the high-quality platform B becomes more attractive than the rival (recall that the former always has a higher market share than the low-quality platform A).

The same conclusion applies under vertical dominance since $\frac{\partial D^{A^*}}{\partial \alpha} = -\frac{q}{3(q-2\alpha)^2} < 0$ and $\frac{\partial D^{B^*}}{\partial \alpha} > 0$.

Regarding the impact of α on equilibrium profit, under horizontal dominance, it is easy to see that the equilibrium profit of the low-quality platform is decreasing with α (since both the price and the market share of the platform decrease with the intensity of the inter-group network effect). In the case of platform B, the two effects are moving in opposite direction but the price effect is dominant and the profit of platform B is also decreasing with α. Thus, it follows that $\frac{\partial \pi^{A^*}}{\partial \alpha} < 0$ and $\frac{\partial \pi^{B^*}}{\partial \alpha} < 0$.

For vertical dominance, the equilibrium profit of the low-quality platform is decreasing with α (since both the price and the market share of the platform decrease with the intensity of the inter-group network effect).

In the case of platform B, the two effects are moving in opposite direction. The dominant effect is, like under horizontal dominance, the price effect and, thus, the profit of platform B is also decreasing with α.

Lemma 26. ***(Impact of the inter-group externality)***

Whatever the type of dominance:

(i) *the intensity of the network effect has the same impact on equilibrium prices and profits;*

(ii) *the profit of both platforms is decreasing with the intensity of the network effect.*

Proof. See Appendix B in section 5.7.2. □

The fact that equilibrium profits are decreasing with α is in line with Armstrong (2006) [9] resulting from the fact that an increase in the intensity of the inter-group network effect intensifies price competition between the two platforms.

5.6 CONCLUSIONS

This paper analyzes price competition between verti-zontally differentiated platforms. In light of this, we extend the pure horizontal differentiation model of two-sided markets presented by Armstrong (2006) [9], also allowing for vertical differentiation between the platforms.

Our verti-zontal differentiation set-up is built on a simplified version of the model proposed by Neven and Thisse (1990) [87]. While the latter allows for quality choice and endogenous location, we take these choices as exogenously given, allowing instead for the existence of inter-group network effects. Our equilibrium analysis shows that equilibrium outcomes depend on the intensity of the inter-group network effects vis-à-vis the gap in the quality of the platforms.

Considering the case in which price discrimination between sides is not allowed, we find that regardless of the intensity of network effects, the high-quality platform always quotes a higher price and attracts a larger fraction of consumers than the low-quality platform. The intuition of this result is that, with a two-dimensional product differentiation, a low-quality platform cannot be a dominant intermediary on the market.

Moreover we find that, under horizontal dominance, the profit of the low-quality platform may decrease as the quality of the better product improves when the quality gap between platforms is sufficiently small. This result contrasts with Shaked and Sutton (1982) [105] seminal contribution.

We also conclude that, whatever the type of dominance, the intensity of the network effect has the same impact on equilibrium prices and profits on both platforms and the profit of both platforms is decreasing with the intensity of the network effect, which confirms Armstrong (2006) [9].

Other extensions of this model are worthwhile as well, addressing issues such as (partial) multi-homing, cost asymmetries, the role of imperfect information, dynamic pricing, endogenous locations and qualities, and so on.

5.7 APPENDIX

5.7.1 Appendix A - Demand functions

Consider the case of vertical dominance. The configuration of demand for different price vectors $\left(p^A, p^B\right)$ is illustrated in figure 7. The figure shows that the specification of the demand function, depends on the magnitude of the price gap $p^A - p^B$.

First, notice that from expressions (5.2) and (5.3), follow:

$$\begin{cases} \widetilde{x}_j\left(0\right) = \frac{1}{2} + \frac{p^B - p^A}{2} + \frac{\alpha\left(2D_k^A - 1\right)}{2}; \\ \widetilde{x}_j\left(1\right) = \frac{1}{2} + \frac{p^B - p^A}{2} + \frac{\alpha\left(2D_k^A - 1\right)}{2} - \frac{q}{2}; \\ \widetilde{y}_j\left(0\right) = \frac{1}{q} + \frac{p^B - p^A}{q} + \frac{\alpha\left(2D_k^A - 1\right)}{q}; \\ \widetilde{y}_j\left(1\right) = \frac{1}{q} + \frac{p^B - p^A}{q} + \frac{\alpha\left(2D_k^A - 1\right)}{q} - \frac{2}{q}. \end{cases}$$

In figure 7(a), all consumers prefer to buy from platform B. In that case, the price gap must high enough so that $\widetilde{y}_j\left(0\right) < 0$, or equivalently, $p^A - p^B > 1 - \alpha$. For price vectors $\left(p^A, p^B\right)$ such that $p^A - p^B > 1 - \alpha$, we have $D_j^{A|VD}\left(p^A, p^B\right) = 0$.

In figure 7(b) some consumers start buying from platform A, although the ones located closer to platform B, always prefer this firm, regardless of their willingness to pay for quality, $y \in \left[0, 1\right]$. In that case, the price gap $p^A - p^B$ must be such that $0 \leq \widetilde{y}_j\left(0\right) < 1$ and $\widetilde{y}_j\left(1\right) < 0$. These inequalities impose the follow condition on the price gap in the case of vertical dominance:

$$-(1-\alpha) + \frac{2\alpha}{q} < p^A - p^B \leq 1 - \alpha, \tag{5.6}$$

with $q > 2$ under vertical dominance. When (5.6) holds, the demand specification is given by:

$$\begin{aligned} D_j^{A|VD}\left(p^A, p^B, D^A\right) &= \frac{\widetilde{x}_j(0) \times \widetilde{y}_j(0)}{2} \Leftrightarrow \\ D_j^{A|VD}\left(p^A, p^B, D^A\right) &= \frac{1}{q}\left(\frac{1}{2} + \frac{p^B - p^A}{2} + \alpha D_j^{A|VD} - \frac{\alpha}{2}\right)^2 \Leftrightarrow \\ D_j^{A|VD}\left(p^A, p^B, D^A\right) &= \frac{1}{4q}\left[1 + \alpha\left(2D_j^{A|VD} - 1\right) + p^B - p^A\right]^2. \end{aligned}$$

Solving with respect to $D_j^{A|VD}$ yields:

$$D_j^{A|VD}\left(p^A, p^B\right) = \frac{q - \alpha\left(p^B - p^A + (1-\alpha)\ \right) \pm \sqrt{q[q - 2\alpha(p^B - p^A + 1 - \alpha\ )]}}{2\alpha^2}.$$

For the convex segment of the demand of the low-quality platform, the specification of the demand is given by:

$$D_j^{A|VD}\left(p^A,p^B\right) = \frac{q-\alpha\left(p^B-p^A+(1-\alpha)\ \right)-\sqrt{q[q-2\alpha(p^B-p^A+1-\alpha\ )]}}{2\alpha^2}.$$

In figure 7(c), we observe that consumers with higher willingness to pay for quality prefer platform B over platform A. However, there are consumers in all locations $x \in [0,1]$ participating in platform A. Accordingly, the following conditions must be verified: $\widetilde{y}_j(0) \leq 1$ and $\widetilde{y}_j(1) \geq 0$, leading to the following conditions on the price gap:

$$1+\alpha-q-\tfrac{2\alpha}{q} \leq p^A-p^B \leq -(1-\alpha)+\tfrac{2\alpha}{q}.$$

In that case, the demand specification is given by:

$$D_j^{A|VD}\left(p^A,p^B\right) = \frac{\widetilde{y}_j(0)+\widetilde{y}_j(1)}{2} = \frac{1}{q-2\alpha}\left(p^B-p^A-\alpha\right).$$

In figure 7(d), the price gap p^A-p^B is such that all consumers located closer to firm A, prefer to participate in this platform, regardless of their willingness to pay for quality $y \in [0,1]$. Hence, $\widetilde{y}_j(0) > 1$ and $0 < \widetilde{y}_j(1) \leq 1$, yielding:

$$-q-(1-\alpha) \leq p^A-p^B < 1+\alpha-q-\tfrac{2\alpha}{q}.$$

In that case, the demand specification can be obtained by computing:

$$\begin{array}{rcl} D_j^{A|VD}\left(p^A,p^B,D^A\right) & = & 1-\frac{[1-\widetilde{x}_j(1)]\times[1-\widetilde{y}_j(1)]}{2} \Leftrightarrow \\ D_j^{A|VD}\left(p^A,p^B,D^A\right) & = & 1-\frac{1}{4q}\left\{\left[1+p^A-p^B+\alpha\left(1-D_j^{A|VD}\right)\right]^2-q^2\right\}. \end{array}$$

Solving with respect to $D_j^{A|VD}$ yields:

$$D_j^{A|VD}\left(p^A,p^B\right) = \frac{\alpha\left(p^A-p^B+1+\alpha+q\ \right)-q\pm\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q\ )]}}{2\alpha^2}.$$

For the convex segment of the demand of the low-quality platform, the specification of the demand is given by:

$$D_j^{A|VD}\left(p^A,p^B\right) = \frac{\alpha\left(p^A-p^B+1+\alpha+q\ \right)-q+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q\ )]}}{2\alpha^2}.$$

Finally, in figure $7(e)$ all consumers prefer to participate in platform A, yielding:

$$D_j^{A|VD}\left(p^A,p^B\right)=1.$$

For this to occur, we must have $\widetilde{y}_j(1)>1$, yielding $p^A-p^B<-q-(1-\alpha)$. In light of this, we have that the demand function $D_j^{A|VD}\left(p^A,p^B\right)$ under vertical dominance, corresponds to a piecewise function with the following specification:

$$D_j^{A|VD}\left(p^A,p^B\right)=$$

$$\begin{cases} 0 & if \quad p^A-p^B>1-\alpha; \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{2\alpha^2} & if \quad -(1+\alpha)+\frac{2\alpha}{q}<p^A-p^B\leq 1-\alpha; \\ \frac{1}{q-2\alpha}\left(p^B-p^A-\alpha\right) & if \quad 1+\alpha-q-\frac{2\alpha}{q}\leq p^A-p^B\leq -(1+\alpha)+\frac{2\alpha}{q}; \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}{2\alpha^2} & if \quad -q-(1-\alpha)\leq p^A-p^B<1+\alpha-q-\frac{2\alpha}{q}; \\ 1 & if \quad p^A-p^B<-q-(1-\alpha). \end{cases}$$

Since each side of the market is assumed to be covered by the two firms, platform's B demand function is simply:

$$D_j^{B|VD}\left(p^A,p^B\right)=1-D_j^{A|VD}\left(p^A,p^B\right).$$

The same comments apply, *mutatis mutandis*, when obtaining the specification of the demand function under horizontal dominance.

In particular, the monopoly outcomes occur under exactly the same conditions. Similarly, the convex and concave segments under horizontal dominance are exactly the same as under vertical dominance, although, the corresponding price domain is different. Accordingly, we have that the convex segment:

$$D_j^{A|HD}\left(p^A,p^B\right)=\frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q\left(q-2\alpha(1-\alpha+p^B-p^A)\ \right)}}{2\alpha^2},$$

holds for $1-\alpha-q+\frac{q\alpha}{2}\leq p^A-p^B<1-\alpha$, and the concave segment:

$$D_j^{A|HD}\left(p^A,p^B\right)=\frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q\left[q-2\alpha(1-\alpha-p^B+p^A+q)\ \right]}}{2\alpha^2},$$

holds for $-q-(1-\alpha)\leq p^A-p^B<-(1-\alpha)-\frac{q\alpha}{2}$.

The situation described in figure $7(c)$ for the case of vertical dominance does not arise in the case of horizontal dominance. Instead, when $-(1-\alpha)-\frac{q\alpha}{2}\leq p^A-p^B\leq 1-\alpha-q+\frac{q\alpha}{2}$, the configuration of demand is depicted in figure $8(a)$. In that case, all consumers located closer to platform A participate in this platform (regardless of their willingness to pay for quality) [10]. This is the case, when $\widetilde{y}_j(0) > 1$ and $\widetilde{y}_j(1) < 0$, yielding:

$$-(1-\alpha) - \tfrac{q\alpha}{2} \leq p^A - p^B \leq 1 - \alpha - q + \tfrac{q\alpha}{2}.$$

The expression of demand can be obtained as follows:

$$\begin{aligned} D_j^{A|HD}\left(p^A, p^B\right) &= \tfrac{\widetilde{x}_j(0)+\widetilde{x}_j(1)}{2} \Leftrightarrow \\ D_j^{A|HD}\left(p^A, p^B\right) &= \tfrac{1}{2}\left[1 + \tfrac{p^B - p^A}{1-\alpha} - \tfrac{q}{2(1-\alpha)}\right]. \end{aligned}$$

Accordingly, the demand function $D_j^{A|HD}\left(p^A, p^B\right)$ under horizontal dominance, corresponds to a piecewise function with the following specification:

$D_j^{A|HD}\left(p^A, p^B\right) =$

$$\begin{cases} 0 & if \quad p^A - p^B > 1-\alpha; \\ \frac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{2\alpha^2} & if \quad 1-\alpha-q+\frac{q\alpha}{2} \leq p^A - p^B < 1-\alpha; \\ \frac{1}{2}\left[1+\frac{p^B-p^A}{1-\alpha}-\frac{q}{2(1-\alpha)}\right] & if \quad -(1-\alpha)-\frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha-q+\frac{q\alpha}{2}; \\ \frac{-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}{2\alpha^2} & if \quad -q-(1-\alpha) \leq p^A - p^B < -(1-\alpha)-\frac{q\alpha}{2}; \\ 1 & if \quad p^A - p^B < -q-(1-\alpha), \end{cases} .$$

and $D_j^{B|HD}\left(p^A, p^B\right) = 1 - D_j^{A|HD}\left(p^A, p^B\right).$ □

[10] Recall that in the case of horizontal dominance, consumers attach more importance to horizontal differentiation vis-à-vis vertical differentiation.

5.7.2 Appendix B - Proofs of Propositions and Lemmas

Proof of Lemma 21

Let us consider the profit of platform B :

$$\pi^B\left(p^A,p^B\right)=2p^B D^B(p^A,p^B),$$

with $D^B(p^A,p^B)=D^{B|VD}\left(p^A,p^B\right)$ under vertical dominance, and $D^B(p^A,p^B)=D^{B|HD}\left(p^A,p^B\right)$ under horizontal dominance. Recall that the demand of platform i is a continuous and non-increasing function of p^i, both under horizontal and vertical dominance.

A. proof under Horizontal Dominance for platform B

In the case of horizontal dominance we obtain that the profit function of platform B is given by:

$\pi^{B|HD}\left(p^A,p^B\right)=$

$$\begin{cases} 2p^B,\ p^A-p^B>1-\alpha;\ \text{(branch 5)} \\ 2p^B\left(1-\frac{\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\right]}{2\alpha^2}\right),\quad 1-\alpha-q+\frac{q\alpha}{2}<p^A-p^B\leq 1-\alpha; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(branch 4)} \\ 2p^B\left\{1-\frac{1}{2}\left[1+\frac{p^B-p^A}{1-\alpha}-\frac{q}{2(1-\alpha)}\right]\right\},\quad -(1-\alpha)-\frac{q\alpha}{2}\leq p^A-p^B\leq 1-\alpha-q+\frac{q\alpha}{2}; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(branch 3)} \\ 2p^B\left\{1-\frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{2\alpha^2}\right\},\quad -q-(1-\alpha)\leq p^A-p^B< \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad < -(1-\alpha)-\frac{q\alpha}{2};\ \text{(branch 2)} \\ 0,\ p^A-p^B<-q-(1-\alpha).\ \text{(branch 1)} \end{cases}$$

(i) **Concave and linear segments of the profit function (*branches 3-5 and 1*)**

Under horizontal dominance, $D^{B|HD}\left(p^A,p^B\right)$ is concave and decreasing for $-(1-\alpha)-\frac{q\alpha}{2}\leq p^A-p^B\leq 1-\alpha$ and it is constant for $p^A-p^B>1-\alpha$ and $p^A-p^B<-q-(1-\alpha)$. Since demand in those price domains is concave and non-increasing, the profit function of platform

B is also a concave function of p^B in that price domain[11]. Note that for the concave segment $1-\alpha-q+\frac{q\alpha}{2}<p^A-p^B\leq 1-\alpha$ of the demand of platform B yields:

$$\begin{aligned}
\frac{\partial D^{B|HD}\left(p^A,p^B\right)}{\partial p^B} &= -\frac{-\alpha+\frac{q\alpha}{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}}{2\alpha^2};\\
\frac{\partial^2 D^{B|HD}\left(p^A,p^B\right)}{\partial (p^B)^2} &= -\frac{1}{2q^{\frac{1}{2}}[q-2\alpha(p^B-p^A+1-\alpha)\,]^{\frac{3}{2}}}<0.
\end{aligned}$$

The profit function of platform B is linear at branch 5 and concave at branches 3 and 4. The first derivative of the profit function relatively to its price at branches 4 and 3 are, respectively, given by:

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}=1-\frac{q}{\alpha^2}+\frac{1}{\alpha}-\frac{p^A}{\alpha}+\frac{2p^B}{\alpha}+\frac{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{\alpha^2}-\frac{p^B\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\,]};$$
$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}=\frac{2(1-\alpha)+q+2p^A-4p^B}{2(1-\alpha)}.$$

Evaluating the derivatives at the kink $p^B=p^A-(1-\alpha)+q-\frac{q\alpha}{2}$, follows that the left hand side (thereafter, LHS) derivative (using the profit function expression of branch 4) and the right hand side (thereafter RHS) derivative (using the profit function expression of branch 3) are given by:

$$\begin{aligned}
\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^-} &= 3-q-\frac{q+2p^A}{2(1-\alpha)};\\
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^+} &= 3-q-\frac{q+2p^A}{2(1-\alpha)}.
\end{aligned}$$

In the frontier between the two concave segments of the profit function, the following inequality must hold:

$$\left.\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^-}\geq \left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A-(1-\alpha)+q-\frac{q\alpha}{2}\right]^+}.$$

We verify that this inequality is verified, since the LHS derivative is equal to the RHS derivative at $p^B=p^A-(1-\alpha)+q-\frac{q\alpha}{2}$.

(ii) **Convex segment of the profit function (***branch 2***)**

[11]Note that since $\pi^B\left(p^A,p^B\right)=2p^BD^B(p^A,p^B)$, follows that $\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}=2D^B(p^A,p^B)+2p^B\frac{\partial D^B\left(p^A,p^B\right)}{\partial p^B}$. Thus, $\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial (p^B)^2}=4\frac{\partial D^B\left(p^A,p^B\right)}{\partial p^B}+2p^B\frac{\partial^2 D^B\left(p^A,p^B\right)}{\partial (p^B)^2}$. Since the demand is (i) decreasing and (ii) concave at the considered segment follows that (i) $\frac{\partial D^B\left(p^A,p^B\right)}{\partial p^B}<0$ and (ii) $\frac{\partial^2 D^B\left(p^A,p^B\right)}{\partial (p^B)^2}<0$, respectively. Then, we obtain $\frac{\partial^2 \pi^B\left(p^A,p^B\right)}{\partial (p^B)^2}<0$.

By contrast, the only domain in which the profit function is not concave is:

$$-q-(1-\alpha)\leq p^A-p^B<-(1-\alpha)-\tfrac{q\alpha}{2},$$

where the demand function $D^{B|HD}\left(p^A,p^B\right)$ is strictly convex, with:

$$\frac{\partial^2 D^{B|HD}\left(p^A,p^B\right)}{\partial\left(p^B\right)^2}=\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}}>0$$

for such values of $\left(p^A,p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi^B\left(p^A,p^B\right)=2p^B D^B(p^A,p^B)=2p^B\left\{1-\frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{2\alpha^2}\right\},$$

The derivative relatively to price is given by:

$$\frac{\partial\pi^B\left(p^A,p^B\right)}{\partial p^B}=\frac{q-q\alpha-\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}-\alpha\left(1-\alpha-2p^B+p^A+\frac{qp^B}{\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}\right)}{\alpha^2} \tag{5.7}$$

Alternatively, expression (5.7) can be re-written as:

$$\frac{\partial\pi^B\left(p^A,p^B\right)}{\partial p^B}=1+\frac{q}{\alpha^2}-\frac{1}{\alpha}-\frac{p^A}{\alpha}-\frac{q}{\alpha}+\frac{2p^B}{\alpha}-\frac{p^B\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}{\alpha[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}-\frac{\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}{\alpha^2}. \tag{5.8}$$

The second derivative of π^B relatively to p^B in the mentioned price domain is given by:

$$\frac{\partial^2\pi^B\left(p^A,p^B\right)}{\partial\left(p^B\right)^2}=\frac{2+\frac{q^2p^B\alpha}{\left(q-2\alpha(1-\alpha-p^B+p^A+q)\ \right)^{\frac{3}{2}}}-\frac{2q}{\sqrt{q\left(q-2\alpha(1-\alpha-p^B+p^A+q)\ \right)}}}{\alpha^2},$$

and the third derivative of the profit function with respect to their price is given by:

$$\frac{\partial^3\pi^B\left(p^A,p^B\right)}{\partial\left(p^B\right)^3}=\frac{3q^3\left[q-2\alpha\left(1-\alpha-\frac{p^B}{2}+p^A+q\right)\right]}{\left[q\left(q-2\alpha(1-\alpha-p^B+p^A+q)\ \right)\right]^{\frac{5}{2}}}.$$

Thus, it is straightforward that $\frac{\partial^3\pi^B\left(p^A,p^B\right)}{\partial\left(p^B\right)^3}>0$ under Assumption 5.1, so that $\frac{\partial\pi^B\left(p^A,p^B\right)}{\partial p^B}$ is strictly convex in p^B for the price domain $-q-(1-\alpha)\leq p^A-p^B<-(1-\alpha)-\frac{q\alpha}{2}$.

Also considering the behavior of $\frac{\partial\pi^B(p^A,p^B)}{\partial p^B}$ in the neighborhood of $p^B = p^A + (1-\alpha) + q$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^B = p^A + (1-\alpha) + q$ in equation (5.7), we obtain:

$$\left.\frac{\partial\pi^B(p^A,p^B)}{\partial p^B}\right|_{p^B=[p^A+(1-\alpha)+q]^-} = -\frac{\left(-q+\sqrt{q^2}\right)\left[q+(1+p^A+q)\alpha-\alpha^2\right]}{q\alpha^2} = 0.$$

Since the LHS derivative of π^B relatively to p^B is null at $p^B = \left[p^A + (1-\alpha) + q\right]^-$ and the second derivative of the profit function evaluated at $p^B = \left[p^A + (1-\alpha) + q\right]^-$ is given by:

$$\left.\frac{\partial^2\pi^B(p^A,p^B)}{\partial (p^B)^2}\right|_{p^B=[p^A+(1-\alpha)+q]^-} = \frac{1-\alpha+p^A+q}{q} > 0,$$

it follows that the profit function $\pi^B\left(p^A,p^B\right)$ reaches to a minimum at the point $p^B = p^A+(1-\alpha)+q$ and, thus, $\frac{\partial\pi^B(p^A,p^B)}{\partial p^B} = 0$ has at most one solution in the domain $-q-(1-\alpha) \leq p^A - p^B < -(1-\alpha) - \frac{q\alpha}{2}$.

(iii) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$, the lower bound of the convex segment of the demand function, the RHS and the LHS derivatives at the kink point $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$ of the profit function of platform B are equal.

(a) To compute the RHS derivative, we plug $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$ in equation (5.7) to obtain:

$$\left.\frac{\partial\pi^B(p^A,p^B)}{\partial (p^B)}\right|_{p^B=\left[p^A+1-\alpha+\frac{q}{2}\alpha\right]^+} = -1 + q - \frac{2p^A+q}{2(1-\alpha)}.$$

(b) To compute the LHS derivative note that, for $p^B = \left[p^A + 1 - \alpha + \frac{q}{2}\alpha\right]^-$, the profit of platform B is equal to:

$$\pi^B\left(p^A,p^B\right) = 2p^B\left\{1 - \frac{1}{2}\left[1 + \frac{p^B-p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right]\right\}.$$

Then, the derivative is given by:

$$\frac{\partial\pi^B(p^A,p^B)}{\partial p^B} = \frac{2-2\alpha+q-4p^B+2p^A}{2(1-\alpha)}.$$

Evaluating the derivative at $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$, we obtain the LHS derivative in the neighborhood of $p^B = \left[p^A + 1 - \alpha + \frac{q}{2}\alpha\right]^-$:

$$\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B}\rceil_{p^B=\left[p^A+1-\alpha+\frac{q}{2}\alpha\right]^-} = -1 + q - \frac{2p^A + q}{2(1-\alpha)}.$$

Thus, the sign of $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}$ is the same on both sides of $p^B = p^A + 1 - \alpha + \frac{q}{2}\alpha$ and both derivatives are equal:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha+\frac{q}{2}\alpha\right]^+} = \frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha+\frac{q}{2}\alpha\right]^-}$$

(iv) **Conclusion**

Thus, combining this result with the fact that π^B is concave on $1-\alpha-q+\frac{q\alpha}{2} \leq p^A - p^B \leq 1-\alpha$, we conclude that the profit function of platform B under horizontal dominance is quasi-concave in the domain of p^B.

B. proof under Vertical Dominance for platform B

In the case of vertical dominance we obtain that the profit function of platform B is given by:

$\pi^{B|VD}\left(p^A, p^B\right) =$

$$\begin{cases} 2p^B,\ p^A - p^B > 1-\alpha;\ \text{(branch 5)} \\ 2p^B\left\{1 - \frac{\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\right]}{2\alpha^2}\right\},\quad -(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha; \\ \qquad\qquad\text{(branch 4)} \\ 2p^B\left[1 - \frac{1}{q-2\alpha}\left(p^B - p^A - \alpha\right)\right],\ 1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q};\ \text{(branch 3)} \\ 2p^B\left\{1 - \frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{2\alpha^2}\right\}, \\ \qquad\qquad -q-(1-\alpha) \leq p^A - p^B < \\ \qquad\qquad < 1+\alpha-q-\frac{2\alpha}{q};\ \text{(branch 2)} \\ 0,\ p^A - p^B < -q-(1-\alpha).\ \text{(branch 1)} \end{cases}$$

(i) **Concave and linear segments of the profit function** *(branches 3-5 and 1)*

As explained in the case of horizontal dominance but, now, under vertical dominance, $D^{B|VD}\left(p^A, p^B\right)$ is concave and decreasing for $1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq 1-\alpha$ and it is constant for $p^A - p^B < -q-(1-\alpha)$. Since demand in those price domains is concave and non-increasing, the profit function of platform B is also a concave function of p^B in that price domain. Note that for the concave segment $-(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha$ of the demand of platform B yields:

$$\begin{aligned}
\frac{\partial D^{B|VD}\left(p^A,p^B\right)}{\partial p^B} &= -\frac{-\alpha+\frac{q\alpha}{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}}{2\alpha^2};\\
\frac{\partial^2 D^{B|VD}\left(p^A,p^B\right)}{\partial (p^B)^2} &= -\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha(p^B-p^A+1-\alpha)\,\right]^{\frac{3}{2}}} < 0.
\end{aligned}$$

The profit function of platform B is linear at branch 5 and concave at branches 3 and 4. The first derivative of the profit function relatively to its price at branches 4 and 3 are, respectively, given by:

$$\begin{gathered}
\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = 1-\frac{q}{\alpha^2}+\frac{1}{\alpha}-\frac{p^A}{\alpha}+\frac{2p^B}{\alpha}+\frac{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{\alpha^2}-\frac{p^B\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\,]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\,]};;\\
\frac{\partial \pi^B\left(p^A,p^B\right)}{\partial p^B} = \frac{2\left(q-\alpha+p^A-2p^B\,\right)}{q-2\alpha}.
\end{gathered}$$

Evaluating the derivatives at $p^B = p^A+1+\alpha-\frac{2\alpha}{q}$, follows that the LHS derivative (using the profit function expression of branch 4) and the RHS derivative (using the profit function expression of branch 3) are given by:

$$\begin{aligned}
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A+1+\alpha-\frac{2\alpha}{q}\right]^-} &= 3-\frac{4}{q}-\frac{2p^A+q}{q-2\alpha};\\
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^B=\left[p^A+1+\alpha-\frac{2\alpha}{q}\right]^+} &= 3-\frac{4}{q}-\frac{2p^A+q}{q-2\alpha}.
\end{aligned}$$

Then, under vertical dominance, the LHS derivative and the RHS derivative at $p^B = p^A+1+\alpha-\frac{2\alpha}{q}$ are equal. Thus, the profit function is strictly concave in the domain $1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq 1-\alpha$.

(ii) **Convex segment of the profit function** *(branch 2)*

By contrast, the only domain in which the profit function is not concave is:

$$-q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\tfrac{2\alpha}{q},$$

where the demand function $D^{B|VD}\left(p^A,p^B\right)$ is strictly convex, with:

$$\frac{\partial^2 D^{B|VD}\left(p^A,p^B\right)}{\partial\left(p^B\right)^2}=\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}}>0$$

for such values of $\left(p^A,p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi^B\left(p^A,p^B\right)=2p^B D^B(p^A,p^B)=2p^B\left\{1-\frac{\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{2\alpha^2}\right\},$$

The derivative relatively to price is given by expression (5.7). The second derivative of π^B relatively to p^B in the mentioned price domain is given by:

$$\frac{\partial^2\pi^B\left(p^A,p^B\right)}{\partial\left(p^B\right)^2}=\frac{2+\frac{q^2p^B\alpha}{[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]^{\frac{3}{2}}}-\frac{2q}{\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}}}{\alpha^2},$$

and the third derivative of the profit function with respect to their price is given by:

$$\frac{\partial^3\pi^B\left(p^A,p^B\right)}{\partial\left(p^B\right)^3}=\frac{3q^3\left[q-2\alpha\left(1-\alpha-\frac{p^B}{2}+p^A+q\right)\right]}{\left\{q\left[q-2\alpha(1-\alpha-p^B+p^A+q)\ \right]\right\}^{\frac{5}{2}}}.$$

Thus, it is straightforward that $\frac{\partial^3\pi^B\left(p^A,p^B\right)}{\partial\left(p^B\right)^3}>0$ under Assumption 5.1 so that $\frac{\partial\pi^B\left(p^A,p^B\right)}{\partial p^B}$ is strictly convex in p^B for the price domain $-q-(1-\alpha)\leq p^A-p^B<1+\alpha-q-\frac{2\alpha}{q}$.

Also considering the behavior of $\frac{\partial\pi^B\left(p^A,p^B\right)}{\partial p^B}$ in the neighborhood of $p^B=p^A+(1-\alpha)+q$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^B=p^A+(1-\alpha)+q$ in equation (5.7), we obtain:

$$\left.\frac{\partial\pi^B\left(p^A,p^B\right)}{\partial p^B}\right|_{p^B=[p^A+(1-\alpha)+q]^-}=-\frac{\left(-q+\sqrt{q^2}\right)\left[q+(1+p^A+q)\alpha-\alpha^2\right]}{q\alpha^2}=0.$$

Since the LHS derivative of π^B relatively to p^B is null at $p^B=\left[p^A+(1-\alpha)+q\right]^-$ and the second derivative of the profit function evaluated at $p^B=\left[p^A+(1-\alpha)+q\right]^-$ is given by:

$$\left.\frac{\partial^2\pi^B\left(p^A,p^B\right)}{\partial\left(p^B\right)^2}\right|_{p^B=[p^A+(1-\alpha)+q]^-}=\frac{1-\alpha+p^A+q}{q}>0,$$

it follows that the profit function $\pi^B\left(p^A,p^B\right)$ reaches to a minimum at the point $p^B=p^A+(1-\alpha)+q$ and, thus, $\frac{\partial\pi^B\left(p^A,p^B\right)}{\partial p^B}=0$ has at most one solution in the domain $-q-(1-\alpha)\leq p^A-p^B<1+\alpha-q-\frac{2\alpha}{q}$.

(iii) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^B = p^A - (1+\alpha) + q + \frac{2\alpha}{q}$ (the lower bound of the convex segment of the demand function), the RHS and the LHS derivatives are equal at this kink point of the profit function of platform B.

(a) To compute the RHS derivative, we plug $p^B = p^A - (1+\alpha) + q + \frac{2\alpha}{q}$ in equation (5.7) to obtain:

$$\left.\tfrac{\partial \pi^B\left(p^A, p^B\right)}{\partial \left(p^B\right)}\right|_{p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^+} = -1 + \frac{4}{q} - \frac{2p^A + q}{q - 2\alpha}.$$

(b) To compute the LHS derivative note that, for $p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^-$, the profit of platform B is equal to:

$$\pi^B\left(p^A, p^B\right) = 2p^B \left[1 - \frac{1}{q - 2\alpha}\left(p^B - p^A - \alpha\right)\right].$$

Then, the derivative is given by:

$$\tfrac{\partial \pi^B\left(p^A, p^B\right)}{\partial p^B} = \frac{2(p^A + q - 2p^B - \alpha)}{q - 2\alpha}.$$

Evaluating the derivative at $p^B = p^A - (1+\alpha) + q + \frac{2\alpha}{q}$, we obtain that the LHS derivative in the neighborhood of $p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^-$ is given by:

$$\left.\tfrac{\partial \pi^B\left(p^A, p^B\right)}{\partial p^B}\right|_{p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^-} = -1 + \frac{4}{q} - \frac{2p^A + q}{q - 2\alpha}.$$

Then, both derivatives have the same sign and are equal:

$$\left.\tfrac{\partial \pi^B\left(p^A, p^B\right)}{\partial p^B}\right|_{p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^-} = \left.\tfrac{\partial \pi^B\left(p^A, p^B\right)}{\partial p^B}\right|_{p^B = \left[p^A - (1+\alpha) + q + \frac{2\alpha}{q}\right]^+}.$$

(iv) **Conclusion**

Thus, combining this result with the fact that π^B is concave on $1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq 1-\alpha$, we conclude that the profit function of platform B under vertical dominance is quasi-concave in p^B.

The proof can be repeated, *mutatis mutandis*, under vertical and horizontal dominance for platform A. Let us now consider the profit of platform A :

$$\pi^A\left(p^A, p^B\right) = 2p^A D^A(p^A, p^B),$$

with $D^A(p^A, p^B) = D^{A|VD}\left(p^A, p^B\right)$ under vertical dominance, and $D^A(p^A, p^B) = D^{A|HD}\left(p^A, p^B\right)$ under horizontal dominance. Recall that the demand of platform i is a continuous and non-increasing function of p^i, both under horizontal and vertical dominance.

C. proof under Horizontal Dominance for platform A

In the case of horizontal dominance we obtain that the profit function of platform A is given by:

$\pi^{A|HD}\left(p^A, p^B\right) =$

$$\begin{cases} 0, p^A - p^B > 1-\alpha; \text{ (branch 5)} \\ \frac{p^A\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\right]}{\alpha^2}, 1-\alpha-q+\frac{q\alpha}{2} < p^A - p^B \leq 1-\alpha; \text{ (branch 4)} \\ p^A\left[1+\frac{p^B-p^A}{1-\alpha}-\frac{q}{2(1-\alpha)}\right], -(1-\alpha)-\frac{q\alpha}{2} \leq p^A-p^B \leq 1-\alpha-q+\frac{q\alpha}{2}; \text{ (branch 3)} \\ \frac{p^A\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{\alpha^2}, \ -q-(1-\alpha) \leq p^A-p^B < -(1-\alpha)-\frac{q\alpha}{2}; \\ \hfill \text{(branch 2)} \\ 2p^A, p^A - p^B < -q-(1-\alpha). \text{(branch 1)} \end{cases}$$

(*i*) **Concave and linear segments of the profit function (*branches 1-3 and 5*)**

Furthermore, in the case of horizontal dominance, $D^{A|HD}\left(p^A, p^B\right)$ is concave and decreasing for $p^A - p^B \leq 1-\alpha-q+\frac{q}{2}\alpha$ and it is constant for $p^A - p^B > 1-\alpha$. Since demand in those price domains is concave and non-increasing, the profit function of platform A is also a concave function of p^A in that price domain[12]. Note that for the concave segment $-q-(1-\alpha) \leq p^A - p^B < -(1-\alpha) - \frac{q\alpha}{2}$ of the demand of platform A yields:

$$\begin{aligned}
\frac{\partial D^{A|HD}\left(p^A,p^B\right)}{\partial p^A} &= \frac{1}{2\alpha} - \frac{\sqrt{q\left[q-2\alpha(1-\alpha+p^A-p^B+q)\ \right]}}{2\alpha\left(q-2\alpha(1-\alpha+p^A-p^B+q)\ \right)}; \\
\frac{\partial^2 D^{A|HD}\left(p^A,p^B\right)}{\partial (p^A)^2} &= -\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}} < 0.
\end{aligned}$$

The profit function of platform A is linear at branch 1 and concave at branches 2 and 3. The first derivative of the profit function relatively to its price at branches 2 and 3 are, respectively, given by:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 1+\frac{1}{\alpha}-\frac{q}{\alpha^2}+\frac{2p^A}{\alpha}+\frac{q}{\alpha}-\frac{p^B}{\alpha}+\frac{\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha^2}-\frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]};$$

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = \frac{2(1-\alpha)+2p^B-4p^A-q}{2(1-\alpha)}.$$

Evaluating the derivatives at the kink $p^A = p^B-(1-\alpha)-\frac{q\alpha}{2}$, follows that the LHS derivative (using the profit function expression of branch 2) and the RHS derivative (using the profit function expression of branch 3) are given by:

$$\begin{aligned}
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^-} &= 3-q+\frac{q}{2(1-\alpha)}-\frac{p^B}{1-\alpha}+\frac{1-\alpha+q\alpha}{\alpha^2}; \\
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^+} &= 3-q+\frac{q}{2(1-\alpha)}-\frac{p^B}{1-\alpha}.
\end{aligned}$$

In the frontier between the two concave segments of the profit function, the following inequality must hold:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^-} \geq \left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B-(1-\alpha)-\frac{q\alpha}{2}\right]^+}.$$

By words, the LHS derivative at the kink point $p^A = p^B-(1-\alpha)-\frac{q\alpha}{2}$ must not be lower than the RHS derivative to obtain a concave profit function in the domain $-q-(1-\alpha) \leq$

[12] Note that since $\pi^A\left(p^A,p^B\right) = 2p^A D^A(p^A,p^B)$, follows that $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 2D^A(p^A,p^B) + 2p^A\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A}$. Thus, $\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2} = 4\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A}+2p^A\frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2}$. Since the demand is (i) decreasing and (ii) concave at the considered segment follows that (i) $\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A} < 0$ and (ii) $\frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2} < 0$, respectively. Then, we obtain $\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2} < 0$.

$p^A - p^B \leq 1 - \alpha - q + \frac{q\alpha}{2}$. We verify that this inequality is verified, since the LHS derivative is higher than the RHS derivative at $p^A = p^B - (1-\alpha) - \frac{q\alpha}{2}$ because:

$$1 - \alpha + q\alpha \geq 0,$$

holds under Assumption 5.1.

(ii) **Convex segment of the profit function (***branch 4***)**

By contrast, the only domain in which the profit function is not concave is:

$$1 - \alpha - q + \tfrac{q}{2}\alpha < p^A - p^B \leq 1 - \alpha,$$

where the demand function $D^{A|HD}\left(p^A, p^B\right)$ is strictly convex, with:

$$\frac{\partial^2 D^{A|HD}\left(p^A,p^B\right)}{\partial (p^A)^2} = \frac{1}{2q^{\frac{1}{2}}\left[q - 2\alpha\left(p^B - p^A + 1 - \alpha\right)\ \right]^{\frac{3}{2}}} > 0$$

for such values of $\left(p^A, p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi^A\left(p^A, p^B\right) = 2p^A D^A(p^A, p^B) = p^A\left[\frac{q - \alpha(1-\alpha+p^B-p^A) - \sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2}\right],$$

The derivative relatively to price is given by:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = -\frac{\left(\alpha p^B - q - 2\alpha p^A - \alpha^2 + \alpha\right)\sqrt{\rho} - 2\alpha q p^B + 3\alpha q p^A + q^2 - 2\alpha q(1-\alpha)}{\alpha^2\sqrt{\rho}} \tag{5.9}$$

with $\rho = -2\alpha q\left(p^B - p^A\ \right) + q^2 - 2\alpha(1-\alpha)q$. Alternatively, expression (5.9) can be re-written as:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 1 + \frac{q}{\alpha^2} - \frac{1+p^B-2p^A}{\alpha} - \frac{\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2} - \frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]}. \tag{5.10}$$

The second derivative of π^A relatively to p^A in the mentioned price domain is given by:

$$\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2} = \frac{2}{\alpha} + \frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]^2} - \frac{2\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]},$$

and the third derivative of the profit function with respect to their price is given by:

$$\frac{\partial^3 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^3} = \frac{3\sqrt{q\left[q - 2\alpha(1-\alpha+p^B-p^A)\ \right]}\left\{q - \alpha\left[2(1-\alpha) + 2p^B - p^A\right]\ \right\}}{\left[q - 2\alpha(1-\alpha+p^B-p^A)\ \right]^3}.$$

Thus, it is straightforward that $\frac{\partial^3 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^3} > 0$ under Assumption 5.1, so that $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}$ is strictly convex in p^A for the price domain $1-\alpha-q+\frac{q}{2}\alpha < p^A - p^B \leq 1-\alpha$.

Also considering the behavior of $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}$ in the neighborhood of $p^A = p^B + (1-\alpha)$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^A = p^B + (1-\alpha)$ in equation (5.9), we obtain:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=[p^B+(1-\alpha)]^-} = -\frac{\left[q+\alpha(1-\alpha+p^B)\ \right]\left(\sqrt{q^2}-q\right)}{q\alpha^2} = 0.$$

Since the LHS derivative of π^A relatively to p^A is null at $p^A = \left[p^B+(1-\alpha)\right]^-$ and the second derivative of the profit function evaluated at $p^A = \left[p^B+(1-\alpha)\right]^-$ is given by:

$$\left.\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2}\right|_{p^A=[p^B+(1-\alpha)]^-} = \frac{p^B+1-\alpha}{q} > 0,$$

it follows that the profit function $\pi^A\left(p^A,p^B\right)$ reaches to a minimum at the point $p^A = p^B + (1-\alpha)$ and, thus, $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 0$ has at most one solution in the domain $1-\alpha-q+\frac{q\alpha}{2} < p^A - p^B \leq 1-\alpha$.

(iii) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$, the lower bound of the convex segment of the demand function, the RHS and the LHS derivatives at the kink point $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$ of the profit function of platform A are equal.

(a) To compute the RHS derivative, we plug $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$ in equation (5.9) to obtain:

$$\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^+} = -1 - \frac{p^B}{1-\alpha} + \frac{q}{2}\left(\frac{3-2\alpha}{1-\alpha}\right).$$

(b) To compute the LHS derivative note that, for $\left[p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha\right]^-$, the profit of platform A is equal to:

$$\pi^A\left(p^A,p^B\right) = p^A\left[1 + \frac{p^B-p^A}{1-\alpha} - \frac{q}{2(1-\alpha)}\right].$$

Then, the derivative is given by:

$$\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = \frac{2(1-\alpha)+2p^B-4p^A-q}{2(1-\alpha)}.$$

Evaluating the derivative at $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$, we obtain the LHS derivative in the neighborhood of $p^A = \left[p^B + 1 - \alpha - q + \frac{q}{2}\alpha\right]^-$:

$$\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial p^A}\rceil_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^-} = -1 - \frac{p^B}{1-\alpha} + \frac{q}{2}\left(\frac{3-2\alpha}{1-\alpha}\right).$$

Thus, the sign of $\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial p^A}$ is the same on both sides of $p^A = p^B + 1 - \alpha - q + \frac{q}{2}\alpha$ and both derivatives are equal:

$$\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^+} = \frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\rceil_{p^A=\left[p^B+1-\alpha-q+\frac{q}{2}\alpha\right]^-}$$

(iv) **Conclusion**

Thus, combining this result with the fact that π^A is concave on $-q-(1-\alpha) < p^A - p^B \leq 1 - \alpha - q + \frac{q}{2}\alpha$, we conclude that on the interval $-q-(1-\alpha) \leq p^A - p^B \leq 1-\alpha$ the profit function of platform A under horizontal dominance has a unique maximum with respect to p^A and, therefore, is quasi-concave in p^A.

D. proof under Vertical Dominance for platform A

In the case of vertical dominance we obtain that the profit function of platform A is given by:

$\pi^{A|VD}\left(p^A, p^B\right) =$

$$\begin{cases} 0,\ p^A - p^B > 1-\alpha;\ \text{(branch 5)} \\ \\ \frac{p^A\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\right]}{\alpha^2},\quad -(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha;\ \text{(branch 4)} \\ \\ \frac{2p^A}{q-2\alpha}\left(p^B - p^A - \alpha\right),\ 1+\alpha-q-\frac{2\alpha}{q} \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q};\ \text{(branch 3)} \\ \\ \frac{p^A\left[-q+\alpha(1+\alpha-p^B+p^A+q)+\sqrt{q[q-2\alpha(1-\alpha-p^B+p^A+q)\ ]}\right]}{\alpha^2},\ -q-(1-\alpha) \leq p^A - p^B < 1+\alpha-q-\frac{2\alpha}{q}; \\ \hfill \text{(branch 2)} \\ \\ 2p^A, p^A - p^B < -q-(1-\alpha).\ \text{(branch 1)} \end{cases}$$

(i) **Concave and linear segments of the profit function** *(branches 1-3 and 5)*

As explained in the case of horizontal dominance but, now, under vertical dominance, $D^{A|VD}\left(p^A, p^B\right)$ is concave and decreasing for $p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q}$ and it is constant for

$p^A - p^B > 1-\alpha$. Since demand in those price domains is concave and non-increasing, the profit function of platform A is also a concave function of p^A in that price domain[13].

Note that for the concave segment $-q-(1-\alpha) \leq p^A - p^B < 1+\alpha - q - \frac{2\alpha}{q}$ of the demand of platform A yields:

$$\begin{aligned}
\frac{\partial D^{A|VD}\left(p^A,p^B\right)}{\partial p^A} &= \frac{1}{2\alpha} - \frac{\sqrt{q\left[q-2\alpha(1-\alpha+p^A-p^B+q)\ \right]}}{2\alpha\left[q-2\alpha(1-\alpha+p^A-p^B+q)\ \right]};\\
\frac{\partial^2 D^{A|VD}\left(p^A,p^B\right)}{\partial (p^A)^2} &= -\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}} < 0.
\end{aligned}$$

The profit function of platform A is linear at branch 1 and concave at branches 2 and 3. The first derivative of the profit function relatively to its price at branches 2 and 3 are, respectively, given by:

$$\begin{aligned}
\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} &= 1+\frac{1}{\alpha}-\frac{q}{\alpha^2}+\frac{2p^A}{\alpha}+\frac{q}{\alpha}-\frac{p^B}{\alpha}+\frac{\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha^2}-\frac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^A-p^B+q)\ ]};\\
\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} &= -\frac{2\left(2p^A-p^B+\alpha\right)}{q-2\alpha}.
\end{aligned}$$

Evaluating the derivatives at $p^A = p^B + 1 + \alpha - q - \frac{2\alpha}{q}$, follows that the LHS derivative (using the profit function expression of branch 2) and the RHS derivative (using the profit function expression of branch 3) are given by:

$$\begin{aligned}
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B+1+\alpha-q-\frac{2\alpha}{q}\right]^-} &= 3-\frac{4}{q}+\frac{q}{q-2\alpha}-\frac{2p^B}{q-2\alpha};\\
\left.\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B+1+\alpha-q-\frac{2\alpha}{q}\right]^+} &= 3-\frac{4}{q}+\frac{q}{q-2\alpha}-\frac{2p^B}{q-2\alpha}.
\end{aligned}$$

Then, under vertical dominance, the LHS derivative and the RHS derivative at $p^A = p^B + 1 + \alpha - q - \frac{2\alpha}{q}$ are equal. Thus, the profit function is concave in the domain $-q-(1-\alpha) \leq p^A - p^B \leq -(1+\alpha) + \frac{2\alpha}{q}$.

(ii) **Convex segment of the profit function** *(branch 4)*

[13] Note that since $\pi^A\left(p^A,p^B\right) = 2p^A D^A(p^A,p^B)$, follows that $\frac{\partial \pi^A\left(p^A,p^B\right)}{\partial p^A} = 2D^A(p^A,p^B) + 2p^A\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A}$. Thus, $\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2} = 4\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A} + 2p^A\frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2}$. Since the demand is (i) decreasing and (ii) concave at the considered segment follows that (i) $\frac{\partial D^A\left(p^A,p^B\right)}{\partial p^A} < 0$ and (ii) $\frac{\partial^2 D^A\left(p^A,p^B\right)}{\partial (p^A)^2} < 0$, respectively. Then, we obtain $\frac{\partial^2 \pi^A\left(p^A,p^B\right)}{\partial (p^A)^2} < 0$.

By contrast, the only domain in which the profit function is not concave is:

$$-(1+\alpha)+\tfrac{2\alpha}{q}<p^A-p^B\leq 1-\alpha,$$

where the demand function $D^{A|VD}\left(p^A,p^B\right)$ is strictly convex, with:

$$\tfrac{\partial^2 D^{A|HD}\left(p^A,p^B\right)}{\partial\left(p^A\right)^2}=\frac{1}{2q^{\frac{1}{2}}\left[q-2\alpha\left(p^B-p^A+1-\alpha\right)\ \right]^{\frac{3}{2}}}>0$$

for such values of $\left(p^A,p^B\right)$.

In that price domain, yields that the profit function is given by:

$$\pi_A\left(p^A,p^B\right)=2p^A D^A(p^A,p^B)=p^A\left\{\tfrac{q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha^2}\right\},$$

The derivative relatively to price is given by expression (5.9). The second derivative of π^A relatively to p^A in the mentioned price domain is given by:

$$\tfrac{\partial^2\pi^A\left(p^A,p^B\right)}{\partial\left(p^A\right)^2}=\tfrac{2}{\alpha}+\tfrac{p^A\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]^2}-\tfrac{2\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}}{\alpha[q-2\alpha(1-\alpha+p^B-p^A)\ ]},$$

and the third derivative of the profit function with respect to their price is given by:

$$\tfrac{\partial^3\pi^A\left(p^A,p^B\right)}{\partial\left(p^A\right)^3}=\tfrac{3\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\left\{q-\alpha\left[2(1-\alpha)+2p^B-p^A\right]\ \right\}}{[q-2\alpha(1-\alpha+p^B-p^A)\ ]^3}.$$

Thus, it is straightforward that $\tfrac{\partial^3\pi^A\left(p^A,p^B\right)}{\partial\left(p^A\right)^3}>0$ under Assumption 5.1, so that $\tfrac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}$ is strictly convex in p^A for the price domain $-(1-\alpha)+\tfrac{2\alpha}{q}<p^A-p^B\leq 1-\alpha$.

Also considering the behavior of $\tfrac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}$ in the neighborhood of $p^A=p^B+(1-\alpha)$, which corresponds to the upper bound of the convex segment of the demand, plugging $p^A=p^B+(1-\alpha)$ in equation (5.9), we obtain:

$$\left.\tfrac{\partial\pi^A\left(p^A,p^B\right)}{\partial p^A}\right|_{p^A=\left[p^B+(1-\alpha)\right]^-}=-\frac{\left[q+\alpha(1-\alpha+p^B)\ \right]\left(\sqrt{q^2}-q\right)}{q\alpha^2}=0.$$

Since the LHS derivative of π^A relatively to p^A is null at $p^A=\left[p^B+(1-\alpha)\right]^-$ and the second derivative of the profit function evaluated at $p^A=\left[p^B+(1-\alpha)\right]^-$ is given by:

$$\left.\tfrac{\partial^2\pi^A\left(p^A,p^B\right)}{\partial\left(p^A\right)^2}\right|_{p^A=\left[p^B+(1-\alpha)\right]^-}=\frac{p^B+1-\alpha}{q}>0,$$

it follows that the profit function $\pi^A\left(p^A, p^B\right)$ reaches to a minimum at the point $p^A = p^B + (1-\alpha)$ and, thus, $\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial p^A} = 0$ has at most one solution in the domain $-(1+\alpha)+\frac{2\alpha}{q} < p^A - p^B \leq 1-\alpha$.

(iii) **Sign of the derivatives at the kink between the linear and convex segment of the demand function**

Furthermore, in the neighborhood of $p^A = p^B - (1+\alpha) + \frac{2\alpha}{q}$ (the lower bound of the convex segment of the demand function), the RHS and the LHS derivatives are equal at this kink point of the profit function of platform A.

(a) To compute the RHS derivative, we plug $p^A = p^B - (1+\alpha) + \frac{2\alpha}{q}$ in equation (5.9) to obtain:

$$\left.\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^+} = \frac{4}{q-2\alpha} - \frac{2p^B}{q-2\alpha} + \frac{2\alpha}{q-2\alpha} - \frac{8\alpha}{q(q-2\alpha)}.$$

(b) To compute the LHS derivative note that, for $\left[p^B - (1+\alpha) + \frac{2\alpha}{q}\right]^-$, the profit of platform A is equal to:

$$\pi^A\left(p^A, p^B\right) = \frac{2p^A}{q-2\alpha}\left(p^B - p^A - \alpha\right).$$

Then, the derivative is given by:

$$\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial p^A} = \frac{2p^B}{q-2\alpha} - \frac{4p^A}{q-2\alpha} - \frac{2\alpha}{q-2\alpha}.$$

Evaluating the derivative at $p^A = p^B - (1+\alpha) + \frac{2\alpha}{q}$, we obtain that the LHS derivative in the neighborhood of $p^A = \left[p^B - (1+\alpha) + \frac{2\alpha}{q}\right]^-$ is given by:

$$\left.\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^-} = \frac{4}{q-2\alpha} - \frac{2p^B}{q-2\alpha} + \frac{2\alpha}{q-2\alpha} - \frac{8\alpha}{q(q-2\alpha)}.$$

Then, both derivatives have the same sign and are equal.

$$\left.\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^-} = \left.\frac{\partial \pi^A\left(p^A, p^B\right)}{\partial (p^A)}\right|_{p^A=\left[p^B-(1-\alpha)+\frac{2\alpha}{q}\right]^+},$$

such that the profit function is continuous and decreasing.

(iv) **Conclusion**

Thus, combining this result with the fact that π^A is concave on $-q-(1-\alpha) \leq p^A - p^B \leq -(1+\alpha)+\frac{2\alpha}{q}$, we conclude that on the interval $-q-(1-\alpha) \leq p^A - p^B \leq 1-\alpha$ the profit function of platform A under vertical dominance is quasi-concave and it has a unique maximum with respect to p^A. $\square$

Proof of Proposition 22

As the profit function is quasi-concave, for each type of dominance, the equilibrium prices can be obtained from the first order conditions:

$$\frac{\partial \pi^i\left(p^A,p^B\right)}{\partial p^i} = 0,\ \ i \in \{A, B\}.$$

Recall that, we concentrate on interior solutions in which both platforms are active. Thus, we exclude the pairs of prices $\left(p^A, p^B\right)$ such that:

$$p^A - p^B > 1 - \alpha \vee p^A - p^B < -q - (1 - \alpha).$$

Consider the case of horizontal dominance, arising when $q < 2$. When:

$$-(1-\alpha) - \frac{q\alpha}{2} \leq p^A - p^B \leq 1 - \alpha - q + \frac{q\alpha}{2},$$

the profit of platform A is equal to:

$$2p^A \left\{ \frac{1}{2} \left[1 + \frac{p^B - p^A}{1-\alpha} - \frac{q}{2(1-\alpha)} \right] \right\}$$

while the profit of platform B is given by:

$$2p^B \left\{ 1 - \frac{1}{2} \left[1 + \frac{p^B - p^A}{1-\alpha} - \frac{q}{2(1-\alpha)} \right] \right\}.$$

Solving the corresponding system of first order conditions, we obtain the equilibrium price candidate:

$$p^{A^*} = 1 - \alpha - \tfrac{q}{6};\ p^{B^*} = 1 - \alpha + \tfrac{q}{6},$$

and the equilibrium market shares:

$$D^{A^*} = \tfrac{1}{2}\left[1 - \tfrac{q}{6(1-\alpha)}\right],\ D^{B^*} = \tfrac{1}{2}\left[1 + \tfrac{q}{6(1-\alpha)}\right].$$

Note that in equilibrium:

$$0 \leq D^{A^*} < \frac{1}{2} < D^{B^*} \leq 1.$$

Then, we must ensure in equilibrium:

$$\begin{aligned} D^{A^*} &\geq 0 \Leftrightarrow q \leq 6(1-\alpha); \\ D^{B^*} &\leq 1 \Leftrightarrow q \leq 6(1-\alpha). \end{aligned}$$

It only remains to verify when the equilibrium price candidate verifies the price domain condition:

$$-(1-\alpha) - \tfrac{q\alpha}{2} \leq p^A - p^B \leq 1 - \alpha - q + \tfrac{q\alpha}{2},$$

which holds for the intersection of the following system of inequalities. Computing, we obtain:

$$\begin{cases} -(1-\alpha) - \frac{q\alpha}{2} \leq p^{A*} - p^{B*}; \\ p^{A*} - p^{B*} \leq 1 - \alpha - q + \frac{q\alpha}{2}. \end{cases} \Leftrightarrow \begin{cases} -(1-\alpha) - \frac{q\alpha}{2} \leq -\frac{q}{3}; \\ -\frac{q}{3} \leq 1 - \alpha - q + \frac{q\alpha}{2}. \end{cases} \Leftrightarrow$$
$$\Leftrightarrow q \leq \tfrac{6(1-\alpha)}{2-3\alpha} \cap q \leq \tfrac{6(1-\alpha)}{4-3\alpha} \Leftrightarrow q < \tfrac{6(1-\alpha)}{4-3\alpha}.$$

Note that:

$$6\,(1-\alpha) > \tfrac{6(1-\alpha)}{4-3\alpha} \Leftrightarrow \alpha < 1, \text{ resulting } q \in \left(0, \tfrac{6(1-\alpha)}{4-3\alpha}\right).$$

Therefore, we conclude that $q \in \left(0, \frac{6(1-\alpha)}{4-3\alpha}\right)$, the condition identified in Proposition 22.

Proof of Proposition 23

As the profit function is quasi-concave, for each type of dominance, the equilibrium prices can be obtained from the first order conditions:

$$\tfrac{\partial \pi^i\left(p^A, p^B\right)}{\partial p^i} = 0, \; i \in \{A, B\}.$$

Recall that, we concentrate on interior solutions in which both platforms are active. Thus, we exclude the pairs of prices $\left(p^A, p^B\right)$ such that:

$$p^A - p^B > 1 - \alpha \vee p^A - p^B < -q - (1-\alpha).$$

Focusing only on the linear segment of the profit function under vertical dominance:

$$1 + \alpha - q - \frac{2\alpha}{q} \leq p^A - p^B \leq -(1-\alpha) + \frac{2\alpha}{q},$$

the profit of platform A is equal to:

$$2p^A \left[\frac{1}{q-2\alpha}\left(p^B - p^A - \alpha\right)\right],$$

while the profit of platform B is given by:

$$2p^B \left[1 - \frac{1}{q-2\alpha}\left(p^B - p^A - \alpha\right)\right].$$

Solving the corresponding system of first order conditions, we obtain the equilibrium prices:

$$p^{A^*} = \tfrac{q}{3} - \alpha;\ p^{B^*} = \tfrac{2q}{3} - \alpha,$$

and the equilibrium market shares:

$$D^{A^*} = \tfrac{q-3\alpha}{3(q-2\alpha)},\ D^{B^*} = \tfrac{2q-3\alpha}{3(q-2\alpha)};$$

Note that in equilibrium:

$$0 \leq D^{A^*} < \frac{1}{2} < D^{B^*} \leq 1.$$

Then, we must ensure in equilibrium:

$$D^{A^*} \geq 0 \Leftrightarrow q \geq 3\alpha;$$

$$D^{B^*} \leq 1 \Leftrightarrow q \geq 3\alpha.$$

Focusing only on the linear segment of the profit function under vertical dominance, given the quasi-concavity of the profit function, we obtain that this equilibrium price candidate is always an equilibrium, as long as:

$$1 + \alpha - q - \tfrac{2\alpha}{q} \leq p^A - p^B \leq -(1+\alpha) + \tfrac{2\alpha}{q}. \tag{5.11}$$

First note that:

$$p^A - p^B = \frac{1}{3}q - \alpha - \left(\frac{2}{3}q - \alpha\right) = -\frac{1}{3}q.$$

The two inequalities of (5.11) above require:

$$1 + \alpha - q - \frac{2\alpha}{q} \leq -\frac{1}{3}q \text{ and } -\frac{1}{3}q \leq -(1+\alpha) + \frac{2\alpha}{q}.$$

Solving the first inequality:

$$1 + \alpha - q - \frac{2\alpha}{q} + \frac{1}{3}q \leq 0 \Leftrightarrow -\frac{1}{3}\left(\frac{-3q + 6\alpha - 3q\alpha + 2q^2}{q}\right) \leq 0.$$

The numerator needs to be positive:

$$-3q + 6\alpha - 3q\alpha + 2q^2 \geq 0. \tag{5.12}$$

The second inequality requires:

$$-\frac{1}{3}q \leq -(1+\alpha) + \frac{2\alpha}{q} \Leftrightarrow -\frac{1}{3}q + (1+\alpha) - \frac{2\alpha}{q} \leq 0 \Leftrightarrow -\frac{1}{3}\left(\frac{-3q + 6\alpha - 3q\alpha + q^2}{q}\right) \leq 0.$$

Again, the numerator needs to be positive:

$$-3q + 6\alpha - 3q\alpha + q^2 \geq 0. \tag{5.13a}$$

Comparing inequalities (5.12) and (5.13a), we see that (5.12) always larger than (5.13a) and therefore when the later is accomplished, (5.12) is also verified. Thus, considering the inequality resulting from (5.13a) corresponds to a parabole turned up. The roots of polynomial $-3q + 6\alpha - 3q\alpha + q^2$ are given by:

$$q_{CR1} = \frac{3}{2}\alpha + \frac{1}{2}\sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3} + \frac{3}{2};\ q_{CR2} = \frac{3}{2}\alpha - \frac{1}{2}\sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3} + \frac{3}{2}.$$

The roots are always real numbers. The second root defines a value for q that is smaller than 2 (and, thus, incompatible with the vertical dominance domain):

$$\frac{3}{2}\alpha - \frac{1}{2}\sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3} + \frac{3}{2} < 2 \Leftrightarrow$$

$$\frac{3}{2}\alpha - 2 + \frac{3}{2} < \frac{1}{2}\sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3} \Leftrightarrow 3\alpha - 1 < \sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3}$$

$$(3\alpha - 1)^2 < 3\left(3\alpha^2 - 2\alpha + 3\right) \Leftrightarrow (3\alpha - 1)^2 - 3\left(3\alpha^2 - 2\alpha + 3\right) < 0 \Leftrightarrow$$

$$-8 < 0.$$

Then, the relevant restriction on q to assure the price domain is:

$$q \geq \frac{3}{2}\alpha + \frac{1}{2}\sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3} + \frac{3}{2}. \tag{5.14}$$

Proof of Lemma 25

Under the domain of pure horizontal dominance, the derivatives with respect to the degree of vertical differentiation are given by:

$$\frac{\partial p^{A^*}}{\partial q} = -\frac{1}{6} < 0 \text{ and } \frac{\partial p^{B^*}}{\partial q} = \frac{1}{6} > 0;$$
$$\frac{\partial D^{A^*}}{\partial q} = -\frac{1}{12(1-\alpha)} < 0 \text{ and } \frac{\partial D^{B^*}}{\partial q} = \frac{1}{12(1-\alpha)} > 0;$$
$$\frac{\partial \pi^{A^*}}{\partial q} = \frac{q-6(1-\alpha)}{18} < 0 \text{ for } q < 6\left(1-\alpha\right) \text{ and } \frac{\partial \pi^{B^*}}{\partial q} = \frac{q+6(1-\alpha)}{18} > 0.$$

Since $\frac{6(1-\alpha)}{4-3\alpha} < 6\left(1-\alpha\right)\ \forall\ \alpha < 1$, we conclude that for $q \in \left(0, \frac{6(1-\alpha)}{4-3\alpha}\right)$:

$$\frac{\partial \pi^{A^*}}{\partial q} < 0,\ \frac{\partial \pi^{B^*}}{\partial q} > 0,\ \forall\ \alpha < [0,1)\,.$$

Under the domain of pure vertical dominance, yields:

$$\frac{\partial p^{A^*}}{\partial q} = \frac{1}{3} > 0 \text{ and } \frac{\partial p^{B^*}}{\partial q} = \frac{2}{3} > 0;$$
$$\frac{\partial D^{A^*}}{\partial q} = \frac{\alpha}{3(q-2\alpha)^2} > 0 \text{ and } \frac{\partial D^{B^*}}{\partial q} = -\frac{\alpha}{3(q-2\alpha)^2} < 0;$$

The derivative of the profit of the low-quality platform A with respect to q is given by:

$$\frac{\partial \pi^{A^*}}{\partial q} = \frac{2(q-\alpha)(q-3\alpha)}{9(q-2\alpha)^2}.$$

Since both equilibrium prices and equilibrium market shares are increasing in q, also $\frac{\partial \pi^{A^*}}{\partial q} > 0$ because the profit function is a linear transformation of prices and market shares.

On the high-quality platform B it follows that:

$$\frac{\partial \pi^{B^*}}{\partial q} = \frac{2(2q-5\alpha)(2q-3\alpha)}{3(q-2\alpha)^2} \lesseqgtr 0. \tag{5.15}$$

The denominator of (5.15) is always positive.

The numerator of (5.15) is equivalent to the following polynomial:

$$8q^2 - 32\alpha q + 30\alpha^2. \tag{5.16}$$

The corresponding roots are given by $q_{CR1} = \frac{3}{2}\alpha$ and $q_{CR2} = \frac{5}{2}\alpha$.

The minimum of the polynomial is obtained at $q_{\min} = 2\alpha$ since $\frac{d^2}{dq^2}\left[8q^2 - 32\alpha q + 30\alpha^2\right] = 16 > 0$.

Notice that (5.14) implies $q > 3\alpha$ and, thus, also $q > \frac{5}{2}\alpha$. Then, it is straightforward that $\frac{\partial \pi^{B^*}}{\partial q} > 0$ under pure vertical dominance.

Proof of Lemma 26

Under the region of horizontal dominance, the derivatives of the outcomes with respect to the intensity of the network effect are given by:

$$\begin{aligned}&\tfrac{\partial p^{A^*}}{\partial \alpha} = -1 < 0 \text{ and } \tfrac{\partial p^{B^*}}{\partial \alpha} = -1 < 0;\\ &\tfrac{\partial D^{A^*}}{\partial \alpha} = -\tfrac{q}{12(1-\alpha)^2} < 0 \text{ and } \tfrac{\partial D^{B^*}}{\partial \alpha} = \tfrac{q}{12(1-\alpha)^2} > 0;\\ &\tfrac{\partial \pi^{A^*}}{\partial \alpha} = -1 + \tfrac{q^2}{36(1-\alpha)^2} \text{ and } \tfrac{\partial \pi^{B^*}}{\partial \alpha} = -1 + \tfrac{q^2}{36(1-\alpha)^2}.\end{aligned}$$

The derivatives of π^{A^*} and π^{B^*} relatively to the network effect are equivalent. Since:

$$-1 + \tfrac{q^2}{36(1-\alpha)^2} \Leftrightarrow q^2 - 36(1-\alpha)^2, \tag{5.17}$$

the roots of the polynomial in (5.17) are given by $q_{CR} = \pm 6(1-\alpha)$. Given the sign of the polynomial, both derivatives are negative for the region of horizontal dominance.

Under the region of vertical dominance, follows that:

$$\begin{aligned}&\tfrac{\partial p^{A^*}}{\partial \alpha} = -1 < 0 \text{ and } \tfrac{\partial p^{B^*}}{\partial \alpha} = -1 < 0;\\ &\tfrac{\partial D^{A^*}}{\partial \alpha} = -\tfrac{q}{3(q-2\alpha)^2} < 0 \text{ and } \tfrac{\partial D^{B^*}}{\partial \alpha} = \tfrac{q}{3(q-2\alpha)^2} > 0;\end{aligned}$$

The derivative of the profit of the low-quality platform with respect to α is given by:

$$\frac{\partial \pi^{A^*}}{\partial \alpha} = -1 + \frac{q^2}{9(q-2\alpha)^2}.$$

Since both equilibrium prices and equilibrium market shares are decreasing in α and the profit function is a linear transformation of prices and market shares, also the impact of the inter-group externality on the equilibrium profit is negative ($\frac{\partial \pi^{A^*}}{\partial \alpha} < 0$).

On the high-quality platform B, follows that:

$$\frac{\partial \pi^{B^*}}{\partial \alpha} = \frac{-8q^2 + 36\alpha q - 36\alpha^2}{9(q-2\alpha)^2} = \frac{q^2 - 9(q-2\alpha)^2}{9(q-2\alpha)^2} = -1 + \frac{q^2}{9(q-2\alpha)^2}. \tag{5.18}$$

Since $\frac{\partial \pi^{A^*}}{\partial \alpha} < 0$ under vertical dominance and because $\frac{\partial \pi^{B^*}}{\partial \alpha} = \frac{\partial \pi^{A^*}}{\partial \alpha}$, then $\frac{\partial \pi^{B^*}}{\partial \alpha}$ is negative. Lemma 26 is, now, straightforward. □

5.7.3 Appendix C - Verti-zontal equilibrium details

The green region in figure 13 corresponds to the equilibrium candidate occurring under the circumstances described in figure 7(b). In this situation, the equilibrium occurs in the strictly convex segment of D^A and in the strictly concave segment of D^B. The profit of platform A is given by:

$$\pi^A = \frac{p^A\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\right]}{\alpha^2},$$

while the profit of platform B is given by:

$$\pi^B = 2p^B\left\{1 - \frac{\left[q-\alpha(1-\alpha+p^B-p^A)-\sqrt{q[q-2\alpha(1-\alpha+p^B-p^A)\ ]}\right]}{2\alpha^2}\right\}.$$

Given the quasi-concavity of the profit function, an equilibrium price candidate is always an equilibrium as long as the price domain condition is verified. Under horizontal dominance, the equilibrium is verified for:

$$1-\alpha-q+\tfrac{q\alpha}{2} < p^{A*}-p^{B*} \leq 1-\alpha$$

and under vertical dominance, the equilibrium is verified for:

$$-(1+\alpha)+\tfrac{2\alpha}{q} < p^{A*}-p^{B*} \leq 1-\alpha.$$

The best reply functions of platform A given the price charged by platform B are given by:

$$\begin{cases} p^A\left(p^B\right) = 1-\alpha+p^B; \\ p^A\left(p^B\right) = \frac{-3q+8\alpha\left(1-\alpha+p^B\right)-\sqrt{q}\sqrt{9q-16\alpha(1-\alpha+p^B)}}{16\alpha}; \\ p^A\left(p^B\right) = \frac{-3q+8\alpha\left(1-\alpha+p^B\right)+\sqrt{q}\sqrt{9q-16\alpha(1-\alpha+p^B)}}{16\alpha}. \end{cases}$$

Given an increment on the price of the high-quality platform, the low-quality platform reacts by also increasing its price. Analytically:

$$\frac{dp^A\left(p^B\right)}{dp^B} = 1 \ \cup \ \frac{dp^A\left(p^B\right)}{dp^B} = \frac{1}{2}\left(1 \pm \frac{\sqrt{q}}{\sqrt{9q-16\alpha(1-\alpha+p^B)}}\right).$$

We obtain that $\frac{dp^A\left(p^B\right)}{dp^B} > 0$ in the third equality if and only if $q > 2(1-\alpha+p^B)$, that is, under pure vertical differentiation. In the case of horizontal dominance, the platform A reacts by decreasing its price when p^B increases while under the case of vertical dominance, platform A reacts increasing their price.

From the best reply functions of platform B yield polynomials of degree three relatively to p^A and of degree six relatively to α. Focusing on the polynomial that belongs to the real space $\mathbb{R}$, we obtain:

$$p^B\left(p^A\right) = \frac{3q-16\alpha(1+p^A)+\frac{-9q^2-288q\alpha^2-16\alpha^2(-1+p^A+3\alpha)^2}{\gamma}}{24\alpha}.$$

with $\gamma = \gamma(\alpha, q, p^A)$ representing a polynomial of degree three relatively to p^A and of degree six relatively to α. Because of this polynomial, we are not able to obtain a closed-form solution. Thus, we apply the theorem of Debreu, Glicksberg and Fan (1952) (see, for example, pp. 34 of Fudenberg and Tirole (1991) [52]) to mention that an equilibrium exists for such parameter region corresponding to the verti-zontal equilibrium.

Remark (Verti-zontal equilibrium)

For any combination (α, q), there exists a price equilibrium candidate in the strictly convex segment of D^A and in the strictly concave segment of D^B. The equilibrium candidate must be an equilibrium:

(i) under horizontal dominance, for $\frac{6(1-\alpha)}{4-3\alpha} < q < 2$;

(ii) under vertical dominance for $2 < q < \frac{3}{2}\alpha + \frac{1}{2}\sqrt{3}\sqrt{3\alpha^2 - 2\alpha + 3} + \frac{3}{2}$.

Proof: *First, the existence of a price equilibrium candidate for such domain region is proved in Theorem 24 (and by the proof of Lemma 21 regarding to the convex (concave) segment of the profit function of platform A (platform B) on each type of dominance); secondly, the threshold for each type of dominance results from Proposition 22 and Proposition 23 together with the domains under which horizontal and vertical differentiation are, respectively, defined.*

□

6.0 CONCLUSIONS

In this thesis, we have considered the simultaneous presence of horizontal and vertical differentiation, which are according to Gabszewicz and Wauthy (2012) [57] relevant characteristics present on the majority of the markets.

In Chapter 3, our innovation relies on embracing vertical and horizontal differentiation through a non-uniform distribution of consumers on the Hotelling line. We find that a perturbation (moving from a situation of pure horizontal differentiation where $\mu = 0,5$ to a case of mixed horizontal and vertical differentiation where, for instance, $\mu = 0,499$) that introduces a negligible difference between the consumer density on the left and on the right side of the city disrupts the existence of equilibrium in the model of Armstrong (2006) [9].

The introduction of a non-uniform distribution of consumers in the model of Armstrong leads to a situation where an indifferent consumer will never be located at the middle of the linear city in the market equilibrium. Instead, the indifferent consumer will always be located at the right of the city center. This finding contrasts with the socially optimal outcome, where both platforms operate in the market, but with the high-quality platform capturing a greater market share than in the market equilibrium.

We also study entry to conclude that inter-group externalities make it easier to deter entry of an inferior-quality platform and facilitate the entrance of a superior-quality platform. However, we consider in this analysis that the status before entry corresponds to the equilibrium of the baseline model, i.e., consumers are already in the incumbent platforms. This means we assume that the entrant faces an additional difficulty in attracting consumers. Coordination is, in this sense, assumed to be adverse to the entrant. Indeed, the successful entry of a superior-quality platform or the successful deterrence of an inferior-quality platform may depend on the level of the inter-group externality if we do not assume that coordination is adverse to the entrant because with the presence of network effects it is not guaranteed that the indifferent consumer location coincides with the setting provided by Gabszewicz and Wauthy (2012) [57].

In a Chapter 4, combining again simultaneously vertical and horizontal product differentiation, our contribution provides a justification (besides the asymmetric intensity of the inter-group externality between the sides of the market) for the presence of divide and conquer strategies in equilibrium: the asymmetry on the cost of quality provision of the platform with highest quality between the two sides of the market.

When only the high-quality platform invests in quality provision and assuming that the cost of quality provision is different between the two sides of the market, we find that the high-quality platform should be specialized on investing on the side where the cost of quality provision is lower and the low-quality platform should specialize itself on the side where the high-quality platform has a higher cost of quality provision. Such strategy appears in equilibrium because the low-quality platform may charge an equilibrium price below its marginal cost to the side where the high-quality platform has a lower cost of quality provision.

Finally, in Chapter 5, we embrace the study of vertical and horizontal differentiation but now in a "square" à la Neven and Thisse (1990) [87]. This model enriches the analysis of the platforms' and consumers' behavior in two-sided markets since it introduces concepts that were not captured in the previous chapters: horizontal and vertical dominance. Despite the tractability problems arising in this model, we were able to prove the existence of an interior equilibrium for different quality gap levels under both types of dominance.

This manuscript proves that the payoff functions of the active players on the market are globally quasi-concave in a two-dimensional product differentiation environment with network effects, allowing to apply the theorem of Debreu, Glicksberg and Fan (1952) (see, for example, pp. 34 of Fudenberg and Tirole (1991) [52]) to ensure the existence of an interior pure-strategy Nash equilibrium.

Moreover we find that, under horizontal dominance, the profit of the low-quality platform decreases as the quality of the better product improves. This result contrasts with the seminal contribution of Shaked and Sutton (1982) [105]. We also conclude that, whatever the type of dominance, the intensity of the network effect has the same impact on equilibrium prices and profits and the profit of both platforms is decreasing with the intensity of the network effect, which confirms the findings of Armstrong (2006) [9].

To conclude, we make an effort to provide a deep study of product differentiation in two-sided markets modeling the same topic with three different approaches to enrich the state of art of the mainstream. These findings provide new thoughts on the field, where it is assumed that only the inter-group externalities are relevant to describe such markets. With this thesis, we show that there exist other relevant aspects that also influence two-sided markets which should not be neglected by researchers.

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